NOTES ON CLIFFORD ALGEBRAS, SPIN GROUPS AND TRIALITY

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1. Introduction

These are notes on the subject of the title. They could be though of as a greatly expanded version of the few pages in [FH91] dedicated to this subject, with the help of the original source [Che54]. They include an exposition of the treatment of the structure of real Clifford algebras, following, of course, [ABS64], mostly because of its æsthetic appeal.

I claim no originality whatsoever.

2. Conventions

All fields will have characteristic different from 2. All vector spaces will be finite dimensional. If $v_1, \dots, v_m$ are elements of a vector space V, we denote by $\langle v_1, \dots, v_m \rangle$ the subspace generated by the v_i.

All algebras will be associative, with an identity.

The imaginary unit in $\mathbb{C}$ will be denoted by $\sqrt{-1}$ or by i.

If V is a K-vector space and $q\colon V \to K$ is a quadratic form, we will also denote by $q\colon V \times V \to K$ the associated symmetric bilinear form, so that $q(v) = q(v, v)$. This should not give rise to confusion.

For any two vectors v and w in $\mathbb{R}^n$ or in $\mathbb{C}^n$, we set $\langle v \mid w \rangle = \sum_{i=1}^n v_i w_i$ and $|v|^2 = \langle v \mid v \rangle = \sum_{i=1}^n v_i^2$. The expression $|v|$ is not well-defined for complex vectors, and will not be employed.

Our notations for matrix groups will be standard; we only remark that by $\mathrm{Sp}_n(\mathbb{C})$ we mean the group of invertible linear transformations that preserve the standard alternating form in $\mathbb{C}^{2n}$ (this is ofter denoted by $\mathrm{Sp}_{2n}(\mathbb{C})$).

3. Exterior algebras

Let V be a vector space over K. Recall that for each integer $k \geq 0$ there is a vector space $\bigwedge^k V$, with an alternating k-linear form $V^k \to \bigwedge^k V$, denoted by $(v_1, \dots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k$, which is universal among all alternating k-linear forms. If $e_1, \dots, e_n$ is a basis for V, the elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $i_1, \dots, i_k$ runs over the $\binom{n}{k}$ possible increasing sequences of integers between 1 and n, form a basis for $\bigwedge^k V$.

The direct sum

$$\bigwedge^\bullet V \overset{\text{def}}{=} \bigoplus_{k=0}^{n} \bigwedge^k V$$

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has natural structure of associative, graded and graded-commutative K-algebra, with the product denoted by $(a, b) \mapsto a \wedge b$. This product is determined by the condition

$$(v_1 \wedge \cdots \wedge v_k) \wedge (w_1 \wedge \cdots \wedge w_l) = v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l.$$

Whenever a is a *homogeneous* element of $\bigwedge^\bullet V$, we denote by $|a|$ its degree; and if we use the notation $|a|$ for an element of $\bigwedge^\bullet V$, we are always assuming that a is homogeneous.

There is a natural embedding $V = \bigwedge^1 V \hookrightarrow \bigwedge^\bullet V$. We have $v \wedge v = 0$ and $v \wedge w + w \wedge v = 0$ for any v and w in V.

This embedding has the following universal property.

Proposition 3.1. *Let A be a K-algebra, $\phi\colon V \to A$ a K-linear function such that $\phi(v)^2 = 0$ for any $v \in V$. Then ϕ extends uniquely to a homomorphism of K-algebras $\bigwedge^\bullet V \to A$.*

Proof. We have

$$\begin{aligned} 0 &= \bigl(\phi(v) + \phi(w)\bigr)^2 \\ &= \phi(v)^2 + \phi(v)\phi(w) + \phi(w)\phi(v) + \phi(w)^2 \\ &= \phi(v)\phi(w) + \phi(w)\phi(v) \end{aligned}$$

for any v and w in V. This allows to prove easily that the k-linear form $V^k \to A$ defined by

$$(v_1, \dots, v_k) \longmapsto \phi(v_1) \dots \phi(v_k)$$

is alternating, and so it defined a K-linear map $\bigwedge^k V \to A$ for any $k \geq 0$. By summing up we get the required algebra homomorphism $\bigwedge^\bullet V \to A$; it is obviously unique, because the elements of V generate $\bigwedge^\bullet V$ as a K-algebra. ♠

The exterior algebra has an automorphism and two anti-automorphisms that will be needed. The *main involution* $\epsilon\colon \bigwedge^\bullet V \to \bigwedge^\bullet V$ is the K-linear map that is characterized by the following properties

(a) $\epsilon(1) = 1$,
(b) $\epsilon(v) = -v$ for any $v \in V$, and
(c) $\epsilon(a \wedge b) = \epsilon(a) \wedge \epsilon(b)$ for all a and b in $\bigwedge^\bullet V$.

It is an automorphism of algebras. It is defined by the obvious formula

$$\epsilon(v_1 \wedge \cdots \wedge v_k) = (-1)^k v_1 \wedge \cdots \wedge v_k;$$

hence we have the alternate definition $\epsilon(a) = (-1)^{|a|}$.

We will also use *the main anti-automorphism*, or the *transposition*, $a \mapsto a^{\mathrm{t}}$ of $\bigwedge^\bullet V$. This is the only K-linear map of degree 0 satisfying the conditions

(a) $1^{\mathrm{t}} = 1$,
(b) $v^{\mathrm{t}} = v$ for any $v \in V$, and
(c) $(a \wedge b)^{\mathrm{t}} = b^{\mathrm{t}} \wedge a^{\mathrm{t}}$.

It is defined by the obvious formula

$$\begin{aligned} (v_1 \wedge \cdots \wedge v_k)^{\mathrm{t}} &= v_k \wedge \cdots \wedge v_1 \\ &= (-1)^{\frac{k(k-1)}{2}} v_1 \wedge \cdots \wedge v_k. \end{aligned}$$

From this formula we have the alternate definition

$$a^{\mathrm{t}} = (-1)^{\frac{|a|(|a|-1)}{2}} a.$$

This is an involution: that is, $(a^{\mathrm{t}})^{\mathrm{t}} = a$.

The *conjugation* $a \mapsto \overline{a}$ is the anti-automorphism of $\bigwedge^\bullet V$ defined by the formula

$$\overline{a} = \epsilon(a)^{\mathrm{t}} = \epsilon(a^{\mathrm{t}}).$$

It is characterized by the conditions

(a) $\overline{1} = 1$,
(b) $\overline{v} = -v$ for any $v \in V$, and
(c) $\overline{a \wedge b} = \overline{b} \wedge \overline{a}$.

It is also defined by the formula

$$a^{\mathrm{t}} = (-1)^{\frac{|a|(|a|+1)}{2}} a.$$

Let $V^\vee$ be the dual space of V; we will denote by

$$\langle - \mid - \rangle \colon V^\vee \times V \longrightarrow K$$

the canonical non-degenerate pairing defined by $\langle \xi \mid v \rangle = \xi(v)$. This pairing induces a pairing

$$\langle - \mid - \rangle \colon \bigwedge^k (V^\vee) \times \bigwedge^k V \longrightarrow K$$

defined by the customary formula

$$\langle \xi_1 \wedge \cdots \wedge \xi_k \mid v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle \xi_i \mid v_j \rangle).$$

If $e_1, \ldots, e_n$ is a basis for V, $\epsilon_1, \ldots, \epsilon_n$ the dual basis in $V^\vee$, then one easily checks that the bases $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ and $\{\epsilon_{j_1} \wedge \cdots \wedge \epsilon_{j_k}\}$ are dual with respect to this pairing: hence the pairing is non-degenerate, and we get an isomorphism $\bigwedge^k (V^\vee) \simeq \left(\bigwedge^k V\right)^\vee$.

By summing over all k we also obtain a non-degenerate bilinear pairing

$$\langle - \mid - \rangle \colon \bigwedge^\bullet V^\vee \times \bigwedge^\bullet V \longrightarrow K$$

in which $\bigwedge^l V^\vee$ and $\bigwedge^k V$ are orthogonal when $k \neq l$.

Proposition 3.2. *If $\xi \in V^\vee$, we have a unique K-linear homomorphism $\bigwedge^\bullet V \to \bigwedge^\bullet V$ of degree -1, called the* left contraction by *ξ, denoted by $a \mapsto \xi \vdash a$, with the following properties.*

(a) *If $v \in V$, then $\xi \vdash v = \langle \xi \mid v \rangle$.*
(b) *$\xi \vdash$ is a* left derivation*: that is, if a and b are in $\bigwedge^\bullet V$, we have*

$$\xi \vdash (a \wedge b) = (\xi \vdash a) \wedge b + (-1)^{|a|} a \wedge (\xi \vdash b).$$

Furthermore, this homomorphism has the following properties.

(i) *If $v_1, \ldots, v_k$ are in V, then*

$$\xi \vdash (v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \xi \mid v_i \rangle v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k,$$

where $\widehat{\ }$ denotes an omitted term.

(ii) *$\xi \vdash a$ is linear in ξ.*
(iii) *$\xi \vdash \colon \bigwedge^\bullet V \to \bigwedge^\bullet V$ is the adjoint to K-linear homomorphism $\mathrm{L}_\xi \colon \bigwedge^\bullet V \to \bigwedge^\bullet V$ defined by $\mathrm{L}_\xi(\alpha) = \xi \wedge \alpha$, through the pairing above.*

(iv) $\xi \vdash (\xi \vdash a) = 0$.
(v) *If ξ and η are in $V^\vee$, then* $\xi \vdash (\eta \vdash z) = -\eta \vdash (\xi \vdash a)$.

Proof. By induction on k, it is easy to see that if $\xi \vdash$ has the required properties (a) and (b), then (i) holds.

On the other hand, the sum on the right hand side of the equation in (i) is easily seen to be 0 whenever two of the v_i coincide. Hence there is a unique linear map $\xi \vdash \colon \bigwedge^k V \to \bigwedge^{k-1} V$ satisfying the equality. One can immediately check that the equality

$$\xi \vdash (a \wedge b) = (\xi \vdash a) \wedge b + (-1)^{|a|} a \wedge (\xi \vdash b)$$

holds, by first looking at the case when a and b are decomposable.

Part (ii) is clear.

For part (iii), we need to prove the equality $\langle \alpha \mid \xi \vdash a \rangle = \langle \xi \wedge \alpha \mid a \rangle$ for any $a \in \bigwedge^k V$ and $\alpha \in \bigwedge^{k-1} V^\vee$. Since both sides of the equality are bilinear in a and α, we may assume that a and α are decomposable. Write $a = v_1 \wedge \cdots \wedge v_k$, $\alpha = \xi_2 \wedge \cdots \wedge \xi_k$, and $\xi = \xi_1$. Then we have $\langle \xi \wedge \alpha \mid a \rangle = \det(\langle \xi_i \mid v_i \rangle)$, while

$$\langle \alpha \mid \xi \vdash a \rangle = \sum_{i=1}^{k} (-1)^{i-1} \langle \xi_1 \mid v_i \rangle \langle \xi_2 \wedge \cdots \wedge \xi_k \mid v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k \rangle;$$

and the determinants that appear in the definition of the terms $\langle \xi_2 \wedge \cdots \wedge \xi_k \mid v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k \rangle$ are the determinants of the matrices obtained from the matrix $(\langle \xi_i \mid v_j \rangle)$ by deleting the first row and the i^{th} column. Hence the result follows from the usual rule for computing a determinant by expanding along a row.

The remaining assertions follow immediately from part (iii), and the facts that

$$\mathrm{L}_\xi \circ \mathrm{L}_\xi(a) = \xi \wedge \xi \wedge a = 0$$

and

$$\begin{aligned} \mathrm{L}_\xi \circ \mathrm{L}_\eta(a) &= \xi \wedge \eta \wedge a \\ &= -\eta \wedge \xi \wedge a \\ &= -\mathrm{L}_\eta \circ \mathrm{L}_\xi(a). \end{aligned}$$ ♠

From this we can also define the *right contraction* $\bigwedge^\bullet V \to \bigwedge^\bullet V$, denoted by $a \mapsto a \dashv \xi$, via the formula

$$a \dashv \xi = (\xi \vdash a^{\mathrm{t}})^{\mathrm{t}}.$$

Here are its main properties.

Proposition 3.3.

(a) *If $v \in V$, then* $v \dashv \xi = \langle \xi \mid v \rangle$.
(b) $\xi \vdash$ *is a* right derivation*: that is, if a and b are in $\bigwedge^\bullet V$, we have*

$$(a \wedge b) \dashv \xi = a \wedge (b \dashv \xi) + (-1)^{|b|} (a \dashv \xi) \wedge b.$$

(c) $\xi \vdash a$ *is linear in* ξ.
(d) $(\xi \vdash a)^{\mathrm{t}} = a^{\mathrm{t}} \dashv \xi$.
(e) $\overline{\xi \vdash a} = -\overline{a} \dashv \xi$.
(f) $a \dashv \xi = (-1)^{|a|+1} \xi \vdash a$.
(g) *The function $a \mapsto a \dashv \xi$ is the adjoint to K-linear homomorphism* $\mathrm{R}_\xi \colon \bigwedge^\bullet V \to \bigwedge^\bullet V$ *defined by* $\mathrm{R}_\xi(\alpha) = \alpha \wedge \xi$*, through the pairing above.*
(h) $(a \dashv \xi) \dashv \xi = 0$.

(i) *If* ξ *and* η *are in* $V^\vee$, *then* $(a \dashv \xi) \dashv \eta = -(a \dashv \eta) \dashv \xi$.
(j) *If* ξ *and* η *are in* $V^\vee$, *then* $\xi \vdash (a \dashv \eta) = (\xi \vdash a) \dashv \eta$.

Proof. These are very easy.

Part (d) follows immediately from the definition, and the fact that the transposition is an involution.

Parts (f) and (e) can be verified in several ways: for example, one can assume that a is decomposable, and then proceede by induction on $|a|$.

I will only sketch the proof of part (g), from which the following ones can be easily deduced. First of all, for all $\alpha \in \bigwedge^\bullet V^\vee$ and $a \in \bigwedge^\bullet V$, we have $\langle \alpha^{\mathrm{t}} \mid a^{\mathrm{t}} \rangle = \langle \alpha \mid a \rangle$. To prove this we may assume that α and a are decomposable, in which case it follows from the fact that the transpose of a matrix has the same determinant as the matrix itself. So we have

$$\begin{aligned} \langle \alpha \wedge \xi \mid a \rangle &= \langle (\alpha \wedge \xi)^{\mathrm{t}} \mid a^{\mathrm{t}} \rangle \\ &= \langle \xi \wedge \alpha^{\mathrm{t}} \mid a^{\mathrm{t}} \rangle \\ &= \langle \alpha^{\mathrm{t}} \mid \xi \vdash a^{\mathrm{t}} \rangle \\ &= \langle \alpha^{\mathrm{t}} \mid (a \dashv \xi)^{\mathrm{t}} \rangle \\ &= \langle \alpha \mid a \dashv \xi \rangle, \end{aligned}$$

as claimed. ♠

For later use, we record the following fact.

Lemma 3.4. *Let* $a \in \bigwedge^\bullet V$. *If* $\xi \vdash a = 0$ *for all* $\xi \in V^\vee$, *then* a *is a scalar.*

Proof. Let $e_1, \dots, e_n$ be a basis for V; for each $I \subseteq \{1, \dots, n\}$ set $e_I = e_{i_1} \wedge e_{i_k}$, where we have written $I = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$. Let $\epsilon_1, \dots, \epsilon_n$ be the dual basis of $V^\vee$. Then for each $i = 1, \dots, n$ we have

$$e_i \vdash e_I = \begin{cases} 0 & \text{if } i \notin I \\ \pm e_{I \setminus \{i\}} & \text{if } i \in I. \end{cases}$$

The elements e_I for a basis for $\bigwedge^\bullet V$; suppose that $\xi \vdash a = 0$ for all $\xi \in V^\vee$, and write $a = \sum_I \alpha_I e_I$. Then

$$0 = \epsilon_i \vdash a = \sum_{i \in I} \pm \alpha_I e_{I \setminus \{i\}}$$

so $\alpha_I = 0$ whenever $i \in I$. Since this holds for every i, we have $\alpha_I = 0$ whenever I is not empty, and a is a multiple of $e_\emptyset = 1$, that is, a scalar. ♠

If $q \colon V \to K$ a quadratic form, for each $v \in V$ and $a \in \bigwedge^\bullet V$ we denote by $v \vdash a$ and $a \dashv v$ the left and right contractions of a by the linear form $q(v, -)$. Let us record the properties of these operations that we are going to use in the future.

Proposition 3.5.

(a) *If* $x \in V$, *then* $v \vdash x = x \dashv v = q(v, x)$.
(b) $v \vdash$ *is a left derivation,* $\dashv v$ *a right derivation.*
(c) $v \vdash a$ *and* $a \dashv v$ *are linear in* v.
(d) $(v \vdash a)^{\mathrm{t}} = a^{\mathrm{t}} \dashv v$.
(e) $\overline{v \vdash a} = \overline{a} \dashv \overline{v} = -\overline{a} \dashv v$.
(f) $a \dashv v = (-1)^{|a|+1} v \vdash a$.
(g) $v \vdash (v \vdash a) = (a \dashv v) \dashv v = 0$.

(h) *If v and w are in V, then* $v \vdash (w \vdash a) = -w \vdash (v \vdash a)$ *and* $(a \dashv v) \dashv w = -(a \dashv w) \dashv v$.
(i) *If v and w are in V, then* $v \vdash (a \dashv w) = (v \vdash a) \dashv w$.

4. CLIFFORD ALGEBRAS

We will work in the category of quadratic forms: the objects are pairs (V, q), where V is a vector space over K and $q \colon V \to K$ is a quadratic form. The arrows $(V, q) \to (V', q')$ are *isometric maps*, that is, K-linear maps $f \colon V \to V'$ with $q'\bigl(f(v)\bigr) = q(v)$ for any $v \in V$. By the polarization formula, this condition is equivalent to the seemingly stronger requirement that $q'\bigl(f(v), f(w)\bigr) = q(v, w)$ for all v and w in V.

Let V be a vector space over K, $q \colon V \to K$ a quadratic form.

Definition 4.1. A *Clifford algebra* $\mathrm{C}(V, q)$ is a K-algebra over K, with a K-linear homomorphism $J \colon V \to \mathrm{C}(V, q)$, such that

(a) $J(v)^2 = q(v)$ for all $v \in V$, and
(b) $J \colon V \to \mathrm{C}(V, q)$ is universal with respect to K-linear maps with the property above, that is, if A is a K-algebra and $\phi \colon V \to A$ is a K-linear map such that $\phi(v)^2 = 0$ for all $v \in V$, then there exists a unique homomorphism of K-algebras $\overline{\phi} \colon \mathrm{C}(V, q) \to A$ such that $\phi = \overline{\phi} \circ J \colon V \to A$.

Proposition 4.2. *There exists a Clifford algebra* $\mathrm{C}(V, q)$ *for any quadratic form* $q \colon V \to K$.

Proof. In the tensor algebra $\mathrm{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ consider the two-sided ideal I generated by the elements of the form $v \otimes v - q(v)$. From the universal property of the tensor algebra itself, it is straightforward to see that the quotient algebra $\mathrm{T}(V)/I$, with the obvious map $J \colon V = V^{\otimes 1} \to \mathrm{T}(V)/I$, has the required universal property. ♠

From the standard categorical arguments one deduces that $\mathrm{C}(V, q)$ is unique up to a unique isomorphism of K-algebras, and that by choosing a $\mathrm{C}(V, q)$ for every vector space with a quadratic form one gets a functor from the category of quadratic forms to the category of K-algebras. This allows to talk about *the* Clifford algebra $\mathrm{C}(V, q)$.

Proposition 4.3. *The Clifford algebra* $\mathrm{C}(V, q)$ *has a unique* $\mathbb{Z}/2\mathbb{Z}$*-grading in which every element of the form* $J(v)$*, where* $v \in V$*, is odd. This grading is functorial, that is, if* $f \colon (V, q) \to (V', q')$ *is an isometric map, the induced homomorphism of algebras* $\mathrm{C}(V, q) \to \mathrm{C}(V', q')$ *preserve the grading.*

Proof. In the construction of Proposition 4.2, the ideal I is homogeneous with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading. This proves existence.

Uniqueness and functoriality are obvious from the fact that $\mathrm{C}(V, q)$ is generated by the $J(v)$. ♠

We will denote by $\mathrm{C}^{+}(V, q)$ the even part of $\mathrm{C}(V, q)$, $\mathrm{C}^{-}(V, q)$ its odd part. From the fact above, we obtain a functor from the category of quadratic forms into the category of $\mathbb{Z}/2\mathbb{Z}$-graded algebras sending (V, q) into $\mathrm{C}(V, q)$.

Example 4.4. If $q = 0$, then $\mathrm{C}(V, q) = \bigwedge^\bullet V$, with the map $J\colon V \to \bigwedge^\bullet V$ being the embedding $V = \bigwedge^1 V \hookrightarrow \bigwedge^\bullet V$. The $\mathbb{Z}/2\mathbb{Z}$-grading has $\bigwedge^+ V = \bigoplus_k \bigwedge^{2k} V$ as its even part, and $\bigwedge^- V = \bigoplus_k \bigwedge^{2k+1} V$ as its odd part.

This is immediate from Proposition 3.1.

We will see later that the function J is always injective: for this reason, we identify V with its image into $\mathrm{C}(V, q)$, and from now on write v for $J(v)$, thus making the notation much lighter. Some readers (probably not very many) might not like the fact that we have removed the J from the notation before knowing for sure that we may do so, and wonder if this might lead to logical error and circular reasoning. These readers are invited to reintroduce all the missing J, and convince themselves that it is not so.

Notice the following fact. If (V, q) is a quadratic form, $\phi\colon V \to A$ a K-linear map into a K-algebra such that $\phi(v)^2 = q(v)$ for all $v \in V$, we have

$$\begin{aligned}\phi(v)\phi(w) + \phi(w)\phi(v) &= \phi(v+w)^2 - \phi(v)^2 - \phi(w)^2\\ &= q(v+w) - q(v) - q(w)\\ &= 2q(v, w)\end{aligned}$$

for any v and w in V. Conversely, if the equation $\phi(v)\phi(w) + \phi(w)\phi(v) = 2q(v, w)$ is always satisfied, by substituing $v = w$ we get $\phi(v)^2 = q(v)$ for all v. Hence, a Clifford algebra can also be defined as the universal algebra with respect to the condition that $vw + wv = 2q(v, w)$ for any v and w in V. In particular, we obtain the following important fact: in the Clifford algebra, two orthogonal vectors anticommute.

Let $e_1, \ldots, e_n$ be an orthogonal basis for V, and set $\alpha_i = q(e_i)$ for $i = 1, \ldots, n$. If $\phi\colon V \to A$ is K-linear map into a K-algebra, then the expressions $\phi(v)\phi(w) + \phi(w)\phi(v)$ and $q(v, w)$ are both symmetric and bilinear: hence they are equal if and only if they are equal whenever v and w are elements of the basis above. This proves the following.

Proposition 4.5. *The Clifford algebra $\mathrm{C}(V, q)$ is the algebra generated by $e_1, \ldots, e_n$, with the relations $e_i^2 = \alpha_i$ for all i, and $e_i e_j + e_j e_i = 0$ for all $i \neq j$.*

Let I be a subset of $\{1, \ldots, n\}$; write it as $I = \{i_1, \ldots, i_k\}$, write e_I for the element $e_{i_1} \ldots e_{i_k}$ of $\mathrm{C}(V, q)$. From the relations above, it is easy to see that $\mathrm{C}(V, q)$ is generated as a vector space by the elements e_I; hence its dimension as a vector space is at most 2^n (we will see that the e_I are linearly independent, and the dimension is precisely 2^n).

From this we deduce the following.

Proposition 4.6. *Let A be a K-algebra of dimension 2^n as a vector space, with elements $u_1, \ldots, u_n$ such that $u_i^2 = \alpha_i$ for all i, and $u_i u_j + u_j u_i = 0$ for all $i \neq j$. Then the homomorphism of K-algebras $\mathrm{C}(V, q) \to A$ that sends each e_i into u_i is an isomorphism.*

This allows us to construct our first interesting examples.

Examples 4.7. Here $K = \mathbb{R}$. We set $\mathrm{C}_n = \mathrm{C}(\mathbb{R}^n, -|x|^2)$ and $\widetilde{\mathrm{C}}_n = \mathrm{C}(\mathbb{R}^n, |x|^2)$. Let us investigate the structure of C_n and $\widetilde{\mathrm{C}}_n$.

(a) Clearly, $\mathrm{C}_0 = \widetilde{\mathrm{C}}_0 = \mathbb{R}$.

(b) C_1 is generated by one element e_1, with the single relation $e_1^2 = -1$. Hence C_1 is commutative, so $C_1 = \mathbb{R}[x]/(x^2+1) = \mathbb{C}$. The even part is $\mathbb{R}$, while the odd part is the imaginary axis $\mathrm{i}\mathbb{R}$.

(c) Analogously, we have $\widetilde{C}_1 = \mathbb{R}[x]/(x^2-1) = \mathbb{R}\times\mathbb{R}$. The even part is the diagonal $\{(x,x)\}$, while the odd part is the antidiagonal $\{(x,-x)\}$.

(d) C_2 is more interesting. It is generated by two generators e_1 and e_2, with relations $e_1^2 = e_2^2 = -1$ and $e_1e_2 + e_2e_1 = 0$. The quaternion algebra $\mathbb{H}$ has dimension 4, and is generated by two elements i and j, with $\mathrm{i}^2 = \mathrm{j}^2 = -1$, and $\mathrm{ij} = \mathrm{k} = -\mathrm{ji}$; hence there is an isomorphism $C_2 \simeq \mathbb{H}$ that sends e_1 into i and e_2 into j. The even part is $\mathbb{R} + \mathbb{R}\mathrm{k} \simeq \mathbb{C}$, while the odd part is $\mathbb{R}\mathrm{i} + \mathbb{R}\mathrm{j}$.

(e) For $\widetilde{C}_2$ we are looking for a 4-dimensional algebra generated by elements e_1 and e_2, subject to the relations $e_1^2 = e_2^2 = 1$ and $e_1e_2 + e_2e_1 = 0$. This is the matrix algebra $M_2(\mathbb{R})$, taking

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The even part C_2^+ is $\mathbb{R} + \mathbb{R}e_1e_2$, consisting of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Since $(e_1e_2)^2 = -e_1^2e_2^2 = -1$, this is isomorphic to $\mathbb{C}$. Indeed, this is the standard realization of $\mathbb{C}$ as a subalgebra of $M_2(\mathbb{C})$, by considering the action of C by multiplication on $\mathbb{C} = \mathbb{R}^2$.

(f) To identify C_3, we need an algebra of dimension 8 with three generators e_1, e_2 and e_3 that anticommmute, with $e_1^2 = e_2^2 = e_3^2 = -1$. The three imaginary units i, j and k in $\mathbb{H}$ satisfy these relations, but the algebra is too small.

We can take the product $\mathbb{H}\times\mathbb{H}$ as an algebra, with elements $e_1 = (\mathrm{i},-\mathrm{i})$, $e_2 = (\mathrm{j},-\mathrm{j})$ and $e_3 = (\mathrm{k},-\mathrm{k})$. We have $\epsilon_1 \overset{\text{def}}{=} e_2e_3 = (\mathrm{i},\mathrm{i})$, $\epsilon_2 \overset{\text{def}}{=} e_3e_1 = (\mathrm{j},\mathrm{j})$ and $\epsilon_3 \overset{\text{def}}{=} e_1e_2 = (\mathrm{k},\mathrm{k})$. Clearly, 1, e_1, e_2, e_3, ϵ_1, ϵ_2, ϵ_3, and $e_1e_2e_3 = (1,-1)$ form a basis of $\mathbb{H}\times\mathbb{H}$. Hence C_3 is $\mathbb{H}\times\mathbb{H}$; the even part C_3^+ is the subspace generated by 1, ϵ_1, ϵ_2 and ϵ_3: this is the diagonal $\{(x,x)\}$, which is isomorphic to $\mathbb{H}$. The odd part is the antidiagonal $\{(x,-x)\}$.

(g) $\widetilde{C}_3$ is the algebra $M_2(\mathbb{C})$. We take

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}.$$

These elements are easily seen to generate $M_2(\mathbb{C})$; furthermore $e_i^2 = 1$ and $e_ie_j + e_je_i = 0$ for all $i \neq j$.

The even part $\widetilde{C}_3^+$ is $\mathbb{R}\oplus\mathbb{R}e_1e_2\oplus\mathbb{R}e_2e_3\oplus e_1e_3$. Since $(e_1e_2)^2 = -e_1^2e_2^2 = -1$, $(e_2e_3)^2 = -e_2^2e_3^2 = -1$, and $e_1e_2e_2e_3 = e_1e_3$, we have that $\widetilde{C}_3^+$ is isomorphic to $\mathbb{H}$.

Change of base field. The construction of the Clifford algebra is compatible with base change. Let (V,q) be a quadratic form on K, $K \subseteq K'$ a field extension. Then q extends naturally to a quadratic form $q_{K'}$ on $V_{K'} \overset{\text{def}}{=} X \otimes_K K'$, in such a way that

$$q_{K'}(v\otimes\alpha, w\otimes\beta) = q(v,w)\alpha\beta$$

for all $v, w \in V$ and $\alpha, \beta \in K'$.

This quadratic form has the following universal property. Given a quadratic form (V', q') over K', we say that a K-linear map $f\colon V \to V'$ is *isometric* when $q'\big(f(v), f(w)\big) = q(v, w) \in K$ for any $v, w \in V$. Any isometric map $f\colon V \to V'$ extends uniquely to a K'-linear map $f'\colon V_{K'} \to V'$, by the formula $f'(v \otimes \alpha) = f(v)\alpha$; and this map is immediately checked to be a isometric. Hence there is a bijective correspondence between isometric maps $V \to V'$ and isometric maps $V_{K'} \to V'$.

Next, recall that given a K-algebra A, the tensor product $A \otimes_K K'$ has a natural structure of K'-algebra, with the product being given by the rule $(a \otimes \alpha)(b \otimes \beta) = ab \otimes \alpha\beta$. This structure has the property that, given a K'-algebra A' and a homomorphism of K-algebras $A \to A'$ (here A' is considered to be a K-algebra via restriction of scalars), this extends to a unique homomorphism of K'-algebra $A \otimes_K K' \to A'$.

Proposition 4.8. *There is a canonical isomorphism of K'-algebras between $\mathrm{C}(V_{K'}, q_{K'})$ and $\mathrm{C}(V, q) \otimes_K K'$.*

Proof. Consider the natural K'-linear map

$$J' \stackrel{\text{def}}{=} J \otimes \mathrm{id}_{K'}\colon V_K \longrightarrow \mathrm{C}(V, q) \otimes_K K'.$$

First of all, $J'(v')^2 = q_{K'}(v')$ for any $v' \in V_{K'}$. This is equivalent to $J'(v')J'(w') + J'(w')J'(v') = 2q(v', w')$ for all v' and w' in $V_{K'}$; and to check this we may assume that v' and w' are of the form $v \otimes 1$ and $w \otimes 1$ for v and w in V, since vectors of this form generate $V_{K'}$; and then the formula is obvious.

Then we need to check that the homomorphism has the universal property that any $\phi'\colon V_{K'} \to A'$ with $\phi'(v')^2 = q_{K'}(v')$ for all $v' \in V'$ factors uniquely through $\mathrm{C}(v, q) \otimes_K K'$. The restriction $\phi\colon V \to A'$ defined by $\phi(v) = \phi'(v \otimes 1)$ has the property that $\phi(v)^2 = q_{K'}(v \otimes 1) = a(v)$ for all $v \in V$, so we get that ϕ factors throuh a homomorphism of K-algebras $\mathrm{C}(V, q) \to A'$. This extends to a unique homomorphism of K'-algebras $\mathrm{C}(V, q) \otimes K' \to A'$; this gives the required factorization. ♠

This gives a natural embedding $\mathrm{C}(V, q) \hookrightarrow \mathrm{C}(V_{K'}, q_{K'})$.

5. Clifford algebras and exterior algebras

We will denote by (V, q) a quadratic form, $e_1, \ldots, e_n$ an orthogonal basis of V. We will write $\bigwedge^+ V$ and $\bigwedge^- V$ for the even and the odd part of $\bigwedge^\bullet V$; these are respectively $\bigoplus_k \bigwedge^{2k} V$ and $\bigoplus_k \bigwedge^{2k+1} V$.

Theorem 5.1. *There is a structure of $\mathbb{R}$-algebra on $\bigwedge^\bullet V$ such that for any $v \in V$ and $a \in \bigwedge^\bullet V$ we have*

$$va = v \wedge a + v \vdash a \tag{5.1}$$

and

$$av = a \wedge v + a \dashv v. \tag{5.2}$$

The embedding $V = \bigwedge^1 V \hookrightarrow \bigwedge^\bullet V$ makes $\bigwedge^\bullet V$ into a Clifford algebra for (V, q).

This structure is functorial, in the sense that if $f\colon V \to V'$ is an isometric map from (V, q) to (V', q'), then $\bigwedge^\bullet f\colon \bigwedge^\bullet V \to \bigwedge^\bullet V'$ is a homomorphism of algebras.

The even part and odd part of $\bigwedge^\bullet V$ as Clifford algebra are $\bigwedge^+ V$ and $\bigwedge^- V$ respectively.

For any a and b in $\bigwedge^\bullet V$ *we have*

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \quad (ab)^{\mathrm{t}} = b^{\mathrm{t}}a^{\mathrm{t}} \quad \text{and} \quad \overline{ab} = \overline{b}\,\overline{a}.$$

Definition 5.2. The new operation defined in $\bigwedge^\bullet V$ is called the *Clifford product*.

Remark 5.3. If $q = 0$, then the Clifford product concides with the wedge product.

Remark 5.4. Either of the two formulas 5.1 and 5.2 above imply that

$$vw = v \wedge w + q(v, w)$$

for any v and w in V.

Corollary 5.5. *The linear map* $J\colon V \to \mathrm{C}(V, q)$ *is injective, and the dimension of the Clifford algebra* $\mathrm{C}(V, q)$ *is* $2^{\dim V}$. *Hence the elements* $e_{i_1} \dots e_{i_k}$, *where* $(i_1, \dots, i_k)$ *ranges over all strictly increasing sequences of integer between* 1 *and* n, *form a basis for* $\mathrm{C}(V, q)$.

Proof of Theorem 5.1. For clarity, we will reintroduce $J\colon V \to \mathrm{C}(V, q)$ in the notation.

We will show that there is an isomorphism of K-vector spaces $\Phi\colon \mathrm{C}(V, q) \to \bigwedge^\bullet V$ with the properties that $\Phi(1) = 1$, and

$$\Phi\bigl(J(v)a\bigr) = v \wedge \Phi(a) + v \vdash \Phi(a)$$

for any $v \in V$ and $a \in \mathrm{C}(V, q)$. Notice that these two conditions imply that

$$\begin{aligned}
\Phi\bigl(J(v)\bigr) &= \Phi\bigl(J(v)1\bigr) \\
&= v \wedge \Phi(1) + v \vdash \Phi(1) \\
&= v \wedge 1 + v \vdash 1 \\
&= v
\end{aligned}$$

for any $v \in V$. Once we have done this, we can define the product in such a way that Φ becomes an isomorphism of K-algebras, as

$$xy = \Phi\bigl(\Phi^{-1}(x)\Phi^{-1}(y)\bigr).$$

Then we have

$$\begin{aligned}
vx &= \Phi\bigl(\Phi^{-1}(v)\Phi^{-1}(y)\bigr) \\
&= \Phi\bigl(J(v)\Phi^{-1}(y)\bigr) \\
&= v \wedge x + v \vdash x
\end{aligned}$$

for any $v \in V$; that is, equation 5.1 is satisfied. It is also clear that $\bigwedge^\bullet V$ becomes a Clifford algebra, since Φ is an isomorphism.

To construct Φ, consider the map $\psi\colon V \to \mathrm{End}_K(\bigwedge^\bullet V)$ defined by the formula

$$\psi(v)x = v \wedge x + v \vdash x.$$

The map ψ is evidently linear. Also, we have

$$\begin{aligned}
\psi(v)^2 x &= v \wedge (v \wedge x + v \vdash x) + v \vdash (v \wedge x + v \vdash x) \\
&= v \wedge v \wedge x + v \wedge (v \vdash x) + v \vdash (v \wedge x) + v \vdash (v \vdash x) \\
&= v \wedge (v \vdash x) + (v \vdash v) \wedge x - v \wedge (v \vdash x) \\
&= q(v)x;
\end{aligned}$$

hence $\psi(v)^2 = q(v)$, and so there exists a unique homomorphism of K-algebras $\Psi\colon \mathrm{C}(V,q) \to \operatorname{End}_K(\bigwedge^\bullet V)$ such that $\Psi\bigl(J(v)\bigr) = \psi(v)$ for all $v \in V$. We define $\Phi\colon \mathrm{C}(V,q) \to \bigwedge^\bullet V$ by the formula

$$\Phi(a) = \Psi(a)1.$$

Then we have

$$\begin{aligned}\Phi\bigl(J(v)a\bigr) &= \Psi\bigl(J(v)a\bigr)1\\ &= \Psi\bigl(J(v)\bigr)\Psi\bigl(a\bigr)1\\ &= \psi(v)\Phi(a)\\ &= v\wedge\Phi(a) + v\vdash\Phi(a)\end{aligned}$$

as required.

We need to show that Φ is an isomorphism: since $\dim_K \mathrm{C}(V,q) \leq 2^n$ while $\dim_K \bigwedge^\bullet V = 2^n$, it is enough to show that Φ is surjective. Take one of the basis elements $e_{i_1}\wedge\cdots\wedge e_{i_k}$ of $\bigwedge^\bullet V$: it is clearly sufficient to prove that

$$\Phi\bigl(J(e_{i_1})\dots J(e_{i_k})\bigr) = e_{i_1}\wedge\cdots\wedge e_{i_k}:$$

this is easy, by induction on k, because $e_{i_1}\vdash(e_{i_2}\wedge\cdots\wedge e_{i_k}) = 0$.

Let us prove formula 5.2. Since $\mathrm{C}(V,q) = \bigwedge^\bullet V$ is generated by elements of V, we may assume that a is of the form $v_1\dots v_k$, for certain $v_1, \dots, v_k$ in V. We will proceed by induction on k; when $k=0$ then $a=1$ and the formula is trivial. Assume that the formula holds when a is a product of $k-1$ elements. Then we have

$$\begin{aligned}av &= (v_1\dots v_k)v\\ &= v_1(v_2\dots v_k v)\\ &= v_1\wedge\bigl((v_2\dots v_k)\wedge v + (v_2\dots v_k)\dashv v\bigr)\\ &\quad + v_1\vdash\bigl((v_2\dots v_k)\wedge v + (v_2\dots v_k)\dashv v\bigr)\\ &= v_1\wedge(v_2\dots v_k)\wedge v + v_1\wedge\bigl((v_2\dots v_k)\dashv v\bigr)\\ &\quad + \bigl(v_1\vdash(v_2\dots v_k)\bigr)\wedge v + (-1)^{k-1}(v_2\dots v_k)q(v_1,v)\\ &\quad + v_1\vdash\bigl((v_2\dots v_k)\dashv v\bigr)\\ &= v_1\wedge(v_2\dots v_k)\wedge v + \bigl(v_1\wedge(v_2\dots v_k)\bigr)\dashv v\\ &\quad + \bigl(v_1\vdash(v_2\dots v_k)\bigr)\wedge v + \bigl(v_1\vdash(v_2\dots v_k)\bigr)\dashv v\\ &= \bigl(v_1\wedge(v_2\dots v_k) + v_1\wedge(v_2\dots v_k)\bigr)\wedge v\\ &\quad + \bigl(v_1\wedge(v_2\dots v_k) + v_1\wedge(v_2\dots v_k)\bigr)\dashv v\\ &= (v_1\dots v_k)\wedge v + (v_1\dots v_k)\dashv v\\ &= a\wedge v + a\dashv v\end{aligned}$$

as claimed.

Functoriality is clear from the construction.

Let us check that $\bigwedge^+ V$ and $\bigwedge^- V$ are the even and odd part of $\bigwedge^\bullet V$ as a Clifford algebra. The even part is generated as a vector space by the product of an even number of elements of V, and similarly for the odd part: hence it is enough to show that if $v_1, \dots, v_k$ are in V, then $v_1 \dots v_k$ is in $\bigwedge^+ V$ if k is even, or in $\bigwedge^- V$ if k is odd. This is easily done by induction on k from formula 5.1.

The formula $\epsilon(ab) = \epsilon(a)\epsilon(b)$ follows from this last fact.

To prove the formula $(ab)^{\mathrm{t}} = b^{\mathrm{t}} a^{\mathrm{t}}$, first assume that $a = v$ is an element of V. Then

$$\begin{aligned}(vb)^{\mathrm{t}} &= (v \wedge b + v \vdash b)^{\mathrm{t}} \\ &= b^{\mathrm{t}} \wedge v^{\mathrm{t}} + b^{\mathrm{t}} \dashv v \\ &= b^{\mathrm{t}} v^{\mathrm{t}},\end{aligned}$$

as claimed. For the general case, we may assume that a is of the form $v_1 \dots v_k$; and then the proof is by induction on k, in the obvious fashion.

The formula $\overline{ab} = \overline{b}\,\overline{a}$ follows from the previous two. ♠

Lemma 5.6. *Let $v_1, \dots, v_k$ be elements of V that are pairwise orthogonal. Then*

$$v_1 \dots v_k = v_1 \wedge \cdots \wedge v_k$$

in $\bigwedge^\bullet V$.

Proof. Easy, by induction on k from either of the formulas of Theorem 5.1. ♠

From now on we will work with this model of the Clifford algebra, and write $\mathrm{C}(V, q)$ for $\bigwedge^\bullet V$ with this product.

There is an increasing filtration $\mathrm{F}_\bullet \mathrm{C}(V, q)$, where $\mathrm{F}_k \mathrm{C}(V, q)$ is defined as the vector subspace of $\mathrm{C}(V, q)$ generated by the products $v_1 \dots v_k$, where $v_1, \dots, v_k \in V$. Thus $\mathrm{F}_k \mathrm{C}(V, q) = 0$ for $k < 0$, and $\mathrm{F}_k \mathrm{C}(V, q) = \mathrm{C}(V, q)$ for $k \geq n$. Furthermore if $a \in \mathrm{F}_k \mathrm{C}(V, q)$ and $b \in \mathrm{F}_l \mathrm{C}(V, q)$, then $ab \in \mathrm{F}_{k+l} \mathrm{C}(V, q)$; thus the associated graded vector space

$$\bigoplus_{k=0}^{\infty} \mathrm{F}_k \mathrm{C}(V, q) / \mathrm{F}_{k-1} \mathrm{C}(V, q)$$

has a natural structure as a graded algebra.

Proposition 5.7.

(a) $\mathrm{F}_k \mathrm{C}(V, q) = \bigoplus_{i \leq k} \bigwedge^i V$.

(b) *If a and b are in $\mathrm{F}_k \mathrm{C}(V, q)$ and $\mathrm{F}_l \mathrm{C}(V, q)$ respectively, then*

$$ab - a \wedge b \in \mathrm{F}_{k+l-2} \mathrm{C}(V, q).$$

(c) *The graded algebra associated with this filtration is canonically isomorphic to $\bigwedge^\bullet V$.*

Proof. Fix an orthogonal basis $e_1, \dots, e_n$ of V: by the multi-linearity in $v_1, \dots, v_k$ of the product $v_1 \dots v_k$, it is easy to see that $\mathrm{F}_k \mathrm{C}(V, q)$ is the subspace generated by the products $e_{i_1} \dots e_{i_k}$ of at most k of the e_i. By using the relations $e_i^2 = q(e_i)$ and $e_i e_j + e_j e_i = 0$, we see that in fact when two of the indices coincide then the product $e_{i_1} \dots e_{i_k}$ is in $\mathrm{F}_{k-2} \mathrm{C}(V, q)$; while when this does not happen then

$$e_{i_1} \dots e_{i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k}$$

by Lemma 5.6. This allows to prove part (a) easily by induction on k.

Part (b) can also be proved easily by using the basis $e_1, \dots, e_n$. We will use the following alternate method: we may assume that a is of the form $v_1 \dots v_k$.

We proceed by induction on k, the statement being obvious when $k = 0$. Set $u = v_2 \dots v_k$; by induction hypothesis we have $ub - u \wedge b \in \mathrm{F}_{k+l-3}\mathrm{C}(V, q)$. Then

$$\begin{aligned} ab - a \wedge b &= v_1 ub - (v_1 u) \wedge b \\ &= v_1(ub - u \wedge b) + v_1(u \wedge b) - v_1 \wedge u \wedge b - (v_1 \vdash u) \wedge b \\ &= v_1(ub - u \wedge b) + v_1 \vdash (u \wedge b) - (v_1 \vdash u) \wedge b; \end{aligned}$$

and all the terms appearing in the last sum are in $\mathrm{F}_{k+l-2}\mathrm{C}(V, q)$.

Part (c) follows from (a) and (b). ♠

Proposition 5.8. *If K has characteristic 0, then for sequence any $v_1, \dots, v_k$ of elements of V we have*

$$v_1 \wedge \dots \wedge v_k = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) v_{\sigma(1)} \dots v_{\sigma(k)}.$$

Proof. Both sides of the equations are k-linear and alternating; hence it is enough to prove the equality when the v_i are among the e_j. If two of the v_i are equal then both sides are 0; otherwise the v_i are pairwise orthogonal, and the result follows from Lemma 5.6. ♠

Assume that q is non-degenerate. Recall that the Lie algebra $\mathfrak{so}(q) \subseteq \operatorname{End}_K(V)$ of $\mathrm{SO}(q)$ consists of linear maps $A \colon V \to V$ which are skew-symmetric, that is, are such that

$$q(Av, w) + q(v, Aw) = 0$$

for any v and w in V; or equivalently such that $q(Av, v) = 0$ for all $v \in V$. If $e_1, \dots, e_n$ is an orthogonal basis for V, and $\alpha_i = q(e_i)$, then this equivalent to the condition that the matrix (a_{ij}) of A satisfies $a_{ii} = 0$ for all i, and $\alpha_i a_{ij} + \alpha_j a_{ji} = 0$ for all $i \neq j$; hence $\mathfrak{so}(q)$ is a vector space of dimension $n(n-1)/2$.

Theorem 5.9. *Assume that q is non-degenerate. Then $\bigwedge^2 V \subseteq \bigwedge^\bullet V = \mathrm{C}(V, q)$ is a Lie subalgebra of $\mathrm{C}(V, q)$, and is canonically isomorphic to $\mathfrak{so}(q)$.*

Proof. Let us show that $\bigwedge^2 V$ is a Lie subalgebra: it is enough to show that the bracket of two decomposable vectors in $\bigwedge^2 V$ is still in $\bigwedge^2 V$.

Let v_1, v_2, w_1 and w_2 be in V. Then we have

$$\begin{aligned} (v_1 \wedge v_2)(w_1 \wedge w_2) &= \big(v_1 v_2 - q(v_1, v_2)\big)(w_1 \wedge w_2) \\ &= v_1\big(v_2(w_1 \wedge w_2)\big) - q(v_1, v_2)(w_1 \wedge w_2) \\ &= v_1\big(v_2 \wedge w_1 \wedge w_2 + v_2 \vdash (w_1 \wedge w_2)\big) - q(v_1, v_2)(w_1 \wedge w_2) \\ &= v_1\big(v_2 \wedge w_1 \wedge w_2 + q(v_2, w_1)w_2 - q(v_2, w_2)w_1\big) \\ &\quad - q(v_1, v_2)(w_1 \wedge w_2) \\ &= v_1 \wedge v_2 \wedge w_1 \wedge w_2 + q(v_2, w_1)(v_1 \wedge w_2) - q(v_2, w_2)(v_1 \wedge w_1) \\ &\quad + v_1 \vdash (v_2 \wedge w_1 \wedge w_2) + q(v_2, w_1)q(v_1, w_2) - q(v_2, w_2)q(v_1, w_1) \\ &\quad - q(v_1, v_2)(w_1 \wedge w_2) \\ &= v_1 \wedge v_2 \wedge w_1 \wedge w_2 + q(v_2, w_1)(v_1 \wedge w_2) - q(v_2, w_2)(v_1 \wedge w_1) \\ &\quad + q(v_1, v_2)(w_1 \wedge w_2) - q(v_1, w_1)(v_2 \wedge w_2) + q(v_1, w_2)(v_2 \wedge w_1) \\ &\quad + q(v_2, w_1)q(v_1, w_2) - q(v_2, w_2)q(v_1, w_1) \\ &\quad - q(v_1, v_2)(w_1 \wedge w_2) \end{aligned}$$

$$
\begin{aligned}
&= v_1 \wedge v_2 \wedge w_1 \wedge w_2 + q(v_2, w_1)q(v_1, w_2) - q(v_2,w_2)q(v_1, w_1)\\
&\quad - q(v_2,w_2)(v_1 \wedge w_1) + q(v_2, w_1)(v_1 \wedge w_2)\\
&\quad + q(v_1, w_2)(v_2 \wedge w_1) - q(v_1, w_1)(v_2 \wedge w_2).
\end{aligned}
$$

The first two terms of the last expression are left invariant by exchanging $v_1 \wedge v_2$ and $w_1 \wedge w_2$, while the other four terms change sign: so we get a formula for the bracket:

$$
\begin{aligned}
[v_1 \wedge v_2, w_1 \wedge w_2] &= 2\bigl(-q(v_2,w_2)(v_1 \wedge w_1) + q(v_2, w_1)(v_1 \wedge w_2)\\
&\qquad + q(v_1, w_2)(v_2 \wedge w_1) - q(v_1, w_1)(v_2 \wedge w_2)\bigr)\\
&\in \textstyle\bigwedge^2 V.
\end{aligned}
$$

So we see that $\bigwedge^2 V$ is a Lie subalgebra of $\mathrm{C}(V, q)$, and have a formula for the bracket.

Next we need to give an isomorphism $\phi\colon \bigwedge^2 V \to \mathfrak{so}(q)$ of Lie algebras between $\bigwedge^2 V$ and $\mathfrak{so}(q)$. As a linear map, it is defined by the formula

$$
\phi(v_1, v_2)x = 2\bigl(q(v_2, x)v_1 - q(v_1, x)v_2\bigr);
$$

it is easy to check that this function is skew-symmetric with respect to q, by showing that $q\bigl(x, \phi(v_1, v_2)x\bigr)$ is identically 0; it is also obviously alternating in v_1 and V_2, so gives a well defined map $\phi\colon \bigwedge^2 V \to \mathfrak{so}(q)$, as claimed.

If e_i is an orthogonal basis as before, one immediately checks that the $\phi(e_i, e_j)$ with $i < j$ are linearly independent; hence ϕ is injective, and, since $\bigwedge^2 V$ and $\mathfrak{so}(q)$ have the same dimension, ϕ is an isomorphism.

Let us compute the bracket: we have

$$
\begin{aligned}
[\phi(v_1 \wedge v_2), \phi(w_1 \wedge w_2)]x &= 2\phi(v_1 \wedge v_2)\bigl(q(w_2, x)w_1 - q(w_1, x)w_2\bigr)\\
&\quad - 2\phi(w_2 \wedge w_1)\bigl(q(v_2, x)v_1 - q(v_1, x)v_2\bigr)\\
&= 4\bigl(q(v_2, w_1)q(w_2, x)v_1 - q(v_1, w_1)q(w_2, x)v_2\\
&\quad - q(v_2, w_2)q(w_1, x)v_1 + q(v_1, w_2)q(w_1, x)v_2\\
&\quad - q(w_2, v_1)q(v_2, x)w_1 + q(w_1, v_1)q(v_2, x)w_2\\
&\quad + q(w_2, v_2)q(v_1, x)w_1 - q(w_1, v_2)q(v_1, x)w_2\bigr)\\
&= 4\bigl(-q(v_2, w_2)\bigl(q(w_1, x)v_1 - q(v_1, x)w_1\bigr)\\
&\quad + q(v_2, w_1)\bigl(q(w_2, x)v_1 - q(v_2, x)w_2\bigr)\\
&\quad + q(v_1, w_2)\bigl(q(w_1, x)v_2 - q(v_2, x)w_1\bigr)\\
&\quad - q(v_1, w_1)\bigl(q(w_2, x)v_2 - q(v_2, x)w_2\bigr)\bigr)\\
&= 2\bigl(-q(v_2, w_2)\phi(v_1 \wedge w_1) + q(v_2, w_1)\phi(v_1 \wedge w_2)\\
&\quad + q(v_1, w_2)\phi(v_2, w_1) - q(v_1, w_1)\phi(v_2, w_2)\bigr)x\\
&\phi\bigl([v_1 \wedge v_2, w_1 \wedge w_2]\bigr)x,
\end{aligned}
$$

as we were claiming. ♠

6. Real Clifford algebras

Recall that if A and B are algebras over K the tensor product $A \otimes_K B$ has a natural algebra structure, such that $(a \otimes b)(a' \otimes b) = aa' \otimes bb'$ for all a and a' in A, b and b' in B. The natural isomorphism of vector spaces $A \otimes B \simeq B \otimes A$ is also an isomorphism of algebras.

As in Examples 4.7, we set $\mathrm{C}_n = \mathrm{C}(\mathbb{R}^n, -|x|^2)$ and $\widetilde{\mathrm{C}}_n = \mathrm{C}(\mathbb{R}^n, |x|^2)$. We have seen the following equalities: $\mathrm{C}_0 = \widetilde{\mathrm{C}}_1 = \mathbb{R}$, $\mathrm{C}_1 = \mathbb{C}$, $\widetilde{\mathrm{C}}_1 = \mathbb{R} \times \mathbb{R}$, $\mathrm{C}_2 = \mathbb{H}$, $\widetilde{\mathrm{C}}_2 = \mathrm{M}_2(\mathbb{R})$ (we have seen more, but this is what we will need). Let us reconstruct the structure of all Clifford algebras, using the classical method of Atiyah, Bott and Shapiro [ABS64].

Theorem 6.1 (Atiyah, Bott and Shapiro)**.** *For each $n \geq 0$ we have isomorphisms of algebras*

$$\mathrm{C}_n \otimes_{\mathbb{R}} \widetilde{\mathrm{C}}_2 \simeq \widetilde{\mathrm{C}}_{n+2} \quad \textit{and} \quad \widetilde{\mathrm{C}}_n \otimes_{\mathbb{R}} \mathrm{C}_2 \simeq \mathrm{C}_{n+2}$$

Proof. We will follow [ABS64] almost exactly, as the treatment there is not easy to improve upon.

Let us prove the existence of the first isomorphism. We will denote by $\widetilde{e}_1$, ..., $\widetilde{e}_{n+2}$ the canonical basis of $\mathbb{R}^{n+2} \subseteq \widetilde{\mathrm{C}}_{n+2}$, e_1, ..., e_n the canonical basis of $\mathbb{R}^n \subseteq \mathrm{C}_n$, ϵ_1, ϵ_2 the canonical basis of $\mathbb{R}^2 \subseteq \widetilde{\mathrm{C}}_2$. According to Proposition 4.5, $\widetilde{\mathrm{C}}_{n+2}$ is the universal algebra generated by the $\widetilde{e}_i$, with relations $\widetilde{e}_i^2 = 1$ and $\widetilde{e}_i\widetilde{e}_j + \widetilde{e}_j\widetilde{e}_i = 0$.

By a straightforward calculation using the identity $(\epsilon_1\epsilon_2)^2 = -\epsilon_1^2\epsilon_2^2 = -1$, one checks that the elements $e_i \otimes \epsilon_1\epsilon_2$ for $i = 1, \ldots, n$, $1 \otimes \epsilon_1$ and $1 \otimes \epsilon_2$ of $\mathrm{C}_n \otimes \widetilde{\mathrm{C}}_2$ anticommute pairwise and have square 1; hence there is a homomorphism $\widetilde{\mathrm{C}}_{n+2} \to \mathrm{C}_n \otimes_{\mathbb{R}} \widetilde{\mathrm{C}}_2$ sending $\widetilde{e}_i$ into $e_i \otimes \epsilon_1\epsilon_2$ for $i = 1, \ldots, n$, $\widetilde{e}_{n+1}$ into $1 \otimes \epsilon_1$ and $\widetilde{e}_{n+2}$ into $1 \otimes \epsilon_2$. Notice that the elements $-\widetilde{e}_i\widetilde{e}_{n+1}\widetilde{e}_{n+2}$ of $\widetilde{\mathrm{C}}_{n+2}$ map to $e_i \otimes 1$ in $\mathrm{C}_n \otimes_{\mathbb{R}} \widetilde{\mathrm{C}}_2$; hence the image of $\widetilde{\mathrm{C}}_{n+2}$ contains a set of generators of $\mathrm{C}_n \otimes_{\mathbb{R}} \widetilde{\mathrm{C}}_2$, and the homomorphism is surjective. Since both algebras have dimension 2^{n+2}, we see that this is an isomorphism.

The argument for the second isomorphism is completely analogous. ♠

To apply the theorem we need to formulas for certain tensor products; we collect these in the next statement.

Proposition 6.2.

(a) *If A, B and C are K-algebras, we have*

$$(A \times B) \otimes_K C \simeq (A \otimes_K C) \times (B \otimes_K C).$$

(b) *If m and n are positive integers and A is a K-algebra, we have*

$$\mathrm{M}_m(K) \otimes_K \mathrm{M}_n(A) \simeq \mathrm{M}_{mn}(A).$$

(c) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$.
(d) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{M}_2(\mathbb{C})$.
(e) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq \mathrm{M}_4(\mathbb{R})$.

These are all isomorphisms of algebras.

Proof. Part (a) is straightforward.

Let us prove part (b). The tensor product $K^m \otimes_K A^n$ has a natural structure of right A-module, in which the multiplication is given by the rule $(v \otimes x)a = v \otimes xa$.

This module is free of rank mn: if (e_i) is the canonical basis of K^m, and (ϵ_j) is the canonical basis of A^n, the tensor products $e_i \otimes \epsilon_j$ form a basis for $K^m \otimes_K A^n$.

If $\alpha \in \mathrm{M}_m(K) = \mathrm{End}_K(K^m)$ and $\phi \in \mathrm{M}_A = \mathrm{End}_A(A_A)$, then $\alpha \otimes \phi \colon K^m \otimes_K A^n \to K^m \otimes_K A^n$ is endomorphism of the right A-module $K^m \otimes_K A^n$; there is a unique K-linear map

$$\mathrm{M}_m(K) \otimes \mathrm{M}_n(A) \longrightarrow \mathrm{End}_A(K^m \otimes_K A^n) \simeq \mathrm{M}_{mn}(A)$$

that sends $\alpha \otimes \phi$ into $\alpha \otimes \phi$ (the first is an element of a tensor product, the second an endomorphism of a tensor product).

It is easy to check that this is a homomorphism of K-algebras. Also, both $\mathrm{M}_m(K) \otimes \mathrm{M}_n(A)$ and $\mathrm{End}_A(K^m \otimes_K A^n)$ are free left A-modules, and the map above is A-linear. By looking at what happens to the natural basis of $\mathrm{M}_m(K) \otimes \mathrm{M}_n(A)$.

Part (c) is standard and easy: one send $(a, b) \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ into $(ab, a\overline{b}) \in \mathbb{C} \times \mathbb{C}$.

Parts (d) and (e) are standard consequences of the theory of central simple algebras. They can be proved directly as follow.

The field $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\mathrm{i}$ is naturally embedded in $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathrm{i} \oplus \mathbb{R}\mathrm{j} \oplus \mathbb{R}\mathrm{k}$; give $\mathbb{H}$ the structure of right vector space over $\mathbb{C}$ via right multiplication; a basis is given by 1 and j. Each element of $\mathbb{H}$ acts on $\mathbb{H}$ by left multiplication, and this action is $\mathbb{C}$-linear. This yields a homomorphism of $\mathbb{R}$-algebras $\phi \colon \mathbb{H} \to \mathrm{M}_2(\mathbb{C})$, which can be complexified, giving a homomorphism $\phi_{\mathbb{C}} \colon \mathbb{H} \otimes_R \mathbb{C} \to \mathrm{M}_2(\mathbb{C})$ defined by $\phi_{\mathbb{C}}(a \otimes \alpha) = \phi(a)\alpha$; and it is easy to check that this is an isomorphism.

To define the isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{M}_4(\mathbb{R})$, send $a \otimes b$ into the linear endomorphism of $\mathbb{H} = \mathbb{R}^4$ defined by $x \mapsto ax\overline{b}$. We leave it to the reader to check that this is an isomorphism of $\mathbb{R}$-algebras. ♠

This allows us to compute all the C_n and $\widetilde{\mathrm{C}}_n$. Let us start from our knowledge of $\mathrm{C}_1 = \mathbb{C}$, $\widetilde{\mathrm{C}}_2 = \mathbb{R} \times \mathbb{R}$, $\mathrm{C}_2 = \mathbb{H}$ and $\widetilde{\mathrm{C}}_2 = \mathrm{M}_2(\mathbb{R})$. We get

$$\begin{aligned} \mathrm{C}_3 &= \widetilde{\mathrm{C}}_1 \otimes \mathrm{C}_2 \\ &= (\mathbb{R} \times \mathbb{R}) \otimes \mathbb{H} \\ &= \mathbb{H} \times \mathbb{H} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{\mathrm{C}}_3 &= \mathrm{C}_1 \otimes \widetilde{\mathrm{C}}_2 \\ &= \mathbb{C} \otimes \mathrm{M}_2(\mathbb{R}) \\ &= \mathrm{M}_2(\mathbb{C}) \end{aligned}$$

two results that we had already obtained. But we proceed further:

$$\begin{aligned} \mathrm{C}_4 &= \widetilde{\mathrm{C}}_2 \otimes \mathrm{C}_2 \\ &= \mathrm{M}_2(\mathbb{R}) \otimes \mathbb{H} \\ &= \mathrm{M}_2(\mathbb{H}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{\mathrm{C}}_4 &= \mathrm{C}_2 \otimes \widetilde{\mathrm{C}}_2 \\ &= \mathbb{H} \otimes \mathrm{M}_2(\mathbb{R}) \\ &= \mathrm{M}_2(\mathbb{H}); \end{aligned}$$

$$\begin{aligned} \mathrm{C}_5 &= \widetilde{\mathrm{C}}_3 \otimes \mathrm{C}_2 \\ &= \mathrm{M}_2(\mathbb{C}) \otimes \mathbb{H} \\ &= \mathrm{M}_2(\mathbb{R}) \otimes \mathbb{C} \otimes \mathbb{H} \\ &= \mathrm{M}_2(\mathbb{R}) \otimes \mathrm{M}_2(\mathbb{C}) \\ &= \mathrm{M}_4(\mathbb{C}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{\mathrm{C}}_3 &= \mathrm{C}_1 \otimes \widetilde{\mathrm{C}}_2 \\ &= (\mathbb{H} \times \mathbb{H}) \otimes \mathrm{M}_2(\mathbb{R}) \\ &= \mathrm{M}_2(\mathbb{H}) \times_2 (\mathbb{H}); \end{aligned}$$

$$\begin{aligned} C_6 &= \widetilde{C}_4 \otimes C_2 \\ &= M_2(\mathbb{H}) \otimes \mathbb{H} \\ &= M_2(\mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{H} \\ &= M_2(\mathbb{R}) \otimes M_4(\mathbb{R}) \\ &= M_8(\mathbb{R}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{C}_6 &= C_4 \otimes \widetilde{C}_2 \\ &= M_2(\mathbb{H}) \otimes M_2(\mathbb{R}) \\ &= M_4(\mathbb{H}); \end{aligned}$$

and so on.

Proceeding in this way, we construct a table identifying all the Clifford algebras C_n and $\widetilde{C}_n$ for n from 0 to 8.

n	C_n	$\widetilde{C}_n$
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \times \mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H} \times \mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H}) \times M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R}) \times M_8(\mathbb{R})$	$M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

At this point the table becomes periodic.

Theorem 6.3. *If k is a positive integer, we have $C_{8k+n} \simeq M_{16^k}(C_n)$ and $\widetilde{C}_{8k+n} \simeq M_{16^k}(\widetilde{C}_n)$.*

Proof. The result is easily reduced to the case $k = 1$.

We have for each $n \geq 0$

$$\begin{aligned} C_{n+4} &= \widetilde{C}_{n+2} \otimes C_2 \\ &= C_n \otimes \widetilde{C_2} \otimes C_2 \\ &= C_n \otimes C_4, \end{aligned}$$

and

$$\begin{aligned} C_{n+8} &= C_{n+4} \otimes C_4 \\ &= C_n \otimes C_4 \otimes C_4 \\ &= C_n \otimes C_8 \\ &= C_n \otimes M_{16}(\mathbb{R}) \\ &= M_{16}(C_n). \end{aligned}$$

Switching C and $\widetilde{C}$ gives the second isomorphism. ♠

One can also compute the even part C_n^+.

Theorem 6.4. *For each $n > 0$, there is an isomorphism of algebras $C_{n-1} \simeq C_n^+$.*

Proof. For each $i = 1, \dots, n-1$ consider the element $e_i e_n$ of C_n^+. An easy calculation shows that $(e_i e_n)^2 = -1$, and that $e_i e_n$ anticommutes with $e_j e_n$ for $i \neq j$; hence there exists a homomorphism of algebras $C_{n-1} \to C_n^+$ sending e_1 into $e_i e_n$ for each $i = 1, \dots, n-1$. This injective, because it sends a basis of C_{n-1} into a part of a basis of C_n; hence it is an isomorphism, because C_{n-1} and C_n^+ have the same dimension 2^{n-1}. ♠

This methods also apply to complex Clifford algebras. In fact, in this case the two forms $|x|^2$ and $-|x|^2$ are isometric, hence $C(\mathbb{C}^n, |x|^2)$ and $C(\mathbb{C}^n, -|x|^2)$ are isomorphic. So $C(\mathbb{C}^2, |x|^2) = C(\mathbb{C}^2, -|x|^2)$, and from the analogue of Theorem 6.1 we can obtain the following result.

Theorem 6.5. *The algebra $C(\mathbb{C}^n, -|x|^2)$ is isomorphic to $M_{2^m}(\mathbb{C})$ when $n = 2m$, and to $M_{2^m}(\mathbb{C}) \times M_{2^m}(\mathbb{C})$ when $n = 2m + 1$.*

We will prove this again later (see Theorems 8.6 and 8.13). In fact the new proof will be much more informative, as it will identify the simple modules over $C(\mathbb{C}^n, -|x|^2)$.

7. Clifford modules

The results of the previous section, together with Theorem A.10, allows us to compute the possible dimensions of modules over Clifford algebras. This is interesting for several reasons: one of them is that it gives information on the possible number of everywhere linearly independent vector fields on spheres.

Definition 7.1. Let n be a positive integer. The *Radon–Hurwitz* number $K(m)$ is the largest non-negative integer n such that $\mathbb{R}^m$ has a structure of left module over C_n.

Remark 7.2. $\mathbb{R}^m$ is always a module over $C_0 = \mathbb{R}$. It is not obvious that there is an upper bound on the n's such that $\mathbb{R}^m$ has a structure of left module over C_n; however, from the next result it follows that $K(m)$ is well defined, and in fact that $K(m) < m$.

Theorem 7.3. *For each m, there are $K(m)$ vector fields on the sphere $S^{m-1} \subseteq \mathbb{R}^m$ that are linearly independent at every point.*

Remark 7.4. A very deep result of Adams ([Ada62]) states that $K(m)$ is in fact the maximum possible number of everywhere linearly independent vector fields on S^{m-1}.

The theorem is an immediate consequence of the following fact.

Proposition 7.5. *For each m and n, there exists a structure of C_n-module on $\mathbb{R}^m$ if and only if there exist $\sigma_1, \dots, \sigma_n$ in the orthogonal group $O_m(\mathbb{R})$, such that $\langle x \mid \sigma_i(x) \rangle = 0$ for any $x \in \mathbb{R}^m$ and $\langle \sigma_i(x) \mid \sigma_j(x) \rangle = 0$ for any $x \in \mathbb{R}^m$ and any $i \neq j$.*

Proof. A structure of C_n module over $\mathbb{R}^m$ is a homomorphism of $\mathbb{R}$-algebras $C_n \to M_m(\mathbb{R})$. Because of the presentation of C_n by generators and relations, there is a structure of C_n-module on $\mathbb{R}^m$ if and only if there exist $\sigma_1, \dots, \sigma_n$ in M_m such that $\sigma_i^2 = -1$, and $\sigma_i \sigma_j + \sigma_j \sigma_i = 0$ for any $i \neq j$. On the other hand, I claim that if the

σ_i exists, then they is a positive inner product on $\mathbb{R}^m$ with respect to which they are orthogonal, hence they can be made orthogonal with respect to the standard inner product $\langle - \mid - \rangle$ by a change of coordinates. To see this, consider the set Q_n of elements of C_n that are of the form ± 1, or $\pm e_i$ for some $i = 1, \ldots, n$; this is a finite subgroup of the group of units in C_n. A C_n-module structure on $\mathbb{R}^m$ gives a representation of this finite group Q_n on $\mathbb{R}^m$; and then it is well known that there is an invariant positive inner product on $\mathbb{R}^m$. Since the e_i are images in $\mathrm{M}_m(\mathbb{R})$ of elements of Q_m, the result is clear.

So there is a module structure on $\mathbb{R}^m$ if and only if there are orthogonal transformation $\sigma_1, \ldots, \sigma_n \in \mathrm{O}_m(\mathbb{R})$ satisfying the relations $\sigma_i^2 = -1$ and $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$ for $i \neq j$. The following lemma ends the proof.

Lemma 7.6. *Let α and β be in $\mathrm{O}_m(\mathbb{R})$.*
(a) $\alpha^2 = -1$ *if and only if* $\langle x \mid \alpha(x) \rangle = 0$ *for all* $x \in \mathbb{R}^m$.
(b) *If* $\alpha^2 = \beta^2 = -1$, *then* $\alpha\beta + \beta\alpha = 0$ *if and only if* $\langle \alpha(x) \mid \beta(x) \rangle = 0$ *for all* $x \in \mathbb{R}^m$.

Proof. Part (a): for any $\alpha \in \mathrm{M}_m(\mathbb{R})$, the condition $\langle x \mid \alpha(x) \rangle = 0$ for all x is equivalent to $\alpha = -\alpha^{\mathrm{t}}$. Since α is orthogonal, $\alpha^{\mathrm{t}} = \alpha^{-1}$, and the result follows.

Part (b): since $\alpha^2 = -1$ and α is orthogonal, we have

$$\langle \alpha(x) \mid \beta(x) \rangle = -\langle x \mid \alpha\beta(x) \rangle.$$

Because of part (a), $\langle \alpha(x) \mid \beta(x) \rangle = 0$ if and only if $\alpha\beta\alpha\beta = (\alpha\beta)^2 = -1$. By multiplying on the left by α and on the right by β we see that this is equivalent to $\beta\alpha = -\alpha\beta$, as claimed. ♠

Let us compute $\mathrm{K}(m)$. It follows from the results in Section 6 that each C_n is either a matrix algebra over a division algebra, or a product of two copies of a matrix algebra over a division algebra. Then we see from Theorem A.10 that that there are either one or two simple modules over C_n, and that their dimensions as real vector spaces are the same. So we deduce the following. For each $n \geq 0$, denote by $\mathrm{L}(n)$ the least $m > 0$ such that $\mathbb{R}^m$ has a structure of C_n-module.

Proposition 7.7. *For any $m \geq 0$ and any $n \geq 0$, the vector space $\mathbb{R}^m$ has the structure of module over C_n if and only if $\mathrm{L}(n)$ divides m.*

Hence, $\mathrm{K}(m)$ is the largest positive integer n such that $\mathrm{L}(n)$ divides m.

Since we know the structure of C_n we can also compute $\mathrm{L}(n)$. If we write $n = 8k + r$, with $k \geq 0$ and $0 \leq r \leq 7$, then from Theorem 6.3 and Theorem A.10 we know that $\mathrm{L}(n) = 16^k \mathrm{L}(r)$. On the other hand, again from Theorem A.10 we see that

$$\begin{aligned} &\mathrm{L}(0) = 1 \\ &\mathrm{L}(1) = 2 \\ &\mathrm{L}(2) = \mathrm{L}(3) = 4 \\ &\mathrm{L}(4) = \mathrm{L}(5) = \mathrm{L}(6) = \mathrm{L}(7) = 8. \end{aligned}$$

From this we get the following result (we leave the elementary details to the reader).

Theorem 7.8. *Write $m = 16^k 2^l q$, were $0 \leq l \leq 3$ and q is odd. Then*

$$\mathrm{K}(n) = 8k + 2^l - 1.$$

8. The spinor space

In this section we will assume that $K = \mathbb{C}$, and that q is the non-degenerate quadratic form $q(x) = -(x_1^2 + \cdots + x_n^2)$ on $V = \mathbb{C}^n$. We will denote the imaginary unit in $\mathbb{C}$ by $\sqrt{-1}$.

Set $n = 2m$ or $n = 2m + 1$, according to whether n is even or odd. Take W and W' to be two totally isotropic subspaces of V of dimension m such that $W \cap W' = \{0\}$. More explicitly, we can define

$$w_i = \frac{\sqrt{-1}\, e_i + e_{i+m}}{2} \quad \text{and} \quad w'_i = \frac{\sqrt{-1}\, e_i - e_{i+m}}{2}$$

for each $i = 1, \ldots, m$. Then $w_1, \ldots, w_m, w'_1, \ldots, w'_m$ are linearly independent in $\mathbb{C}^n$, and $q(w_i, w_j) = q(w'_i, w'_j) = 0$ for all i and j, $q(w_i, w'_j) = 0$ for $i \neq j$, and $q(w_i, w'_i) = 1/2$. Hence

$$W \stackrel{\text{def}}{=} \langle w_1, \ldots, w_m \rangle \quad \text{and} \quad W' \stackrel{\text{def}}{=} \langle w'_1, \ldots, w'_m \rangle$$

are totally isotropic, and their intersection is $\{0\}$.

When $n = 2m$, then $w_1, \ldots, w_m, w'_1, \ldots, w'_m$ form a basis of V, and $W \oplus W' = V$. When $n = 2m + 1$, we obtain a basis by adding

$$u_0 \stackrel{\text{def}}{=} \sqrt{-1}\, e_{2m+1};$$

furthermore we set

$$U \stackrel{\text{def}}{=} \langle u_0 \rangle = \langle e_{2m+1} \rangle,$$

so that $V = W \oplus W' \oplus U$.

Definition 8.1. The vector space $\bigwedge^\bullet W$ is *the spinor space*. The elements of $\bigwedge^\bullet W$ are called *spinors*.

The point is that the spinor space is a module over $\mathrm{C}(\mathbb{C}^n, q)$. The construction is a little different according to whether n is even or odd.

The even case. Since $\mathbb{C}^n = W \oplus W'$, we will write any element of $\mathbb{C}^n$ in the form $w + w'$, where $w \in W$ and $w' \in W'$.

Theorem 8.2. *There is a unique structure of $\mathrm{C}(\mathbb{C}^n, q)$-module on $\bigwedge^\bullet W$, such that*

$$(w + w')x = w \wedge x + 2w' \vdash x \quad \text{for any } w + w' \in \mathbb{C}^n \subseteq \mathrm{C}(\mathbb{C}^n, q).$$

Proof. Uniqueness is clear from the fact that $\mathbb{C}^n$ generates $\mathrm{C}(\mathbb{C}^n, q)$.

For existence, consider the linear map $\psi \colon W \oplus W' = \mathbb{C}^n \to \mathrm{End}_{\mathbb{C}}\left(\bigwedge^\bullet W\right)$ defined by the formula

$$\psi(w + w')x = w \wedge x + 2w' \vdash x.$$

We have

$$\begin{aligned}
\psi(w + w')^2 x &= \psi(w + w')(w \wedge x + 2w' \vdash x) \\
&= w \wedge w \wedge x + 2w \wedge (w' \vdash x) + w' \vdash (w \wedge x) + 4w' \vdash (w' \vdash x) \\
&= 2w \wedge (w' \vdash x) + 2q(w', w)x - 2w \wedge (w' \vdash x) \\
&= q(w + w')x;
\end{aligned}$$

hence $\psi(w + w')^2 = q(w + w')$, and ψ extends to a homomorphism of $\mathbb{C}$-algebras $\mathrm{C}(\mathbb{C}^n, q) \to \mathrm{End}_{\mathbb{C}}\left(\bigwedge^\bullet \mathbb{C}^n\right)$, which defines the required module structure. ♠

We refer to $\bigwedge^\bullet W$ with this $\mathrm{C}(\mathbb{C}^n, q)$-module structure as *the spinor module*.

Here is an alternate construction of this module structure; it will not be used in what follows.

The Clifford products in $\bigwedge^\bullet W$ and $\bigwedge^\bullet W'$ coincide with the wedge products, since W and W' are totally isotropic; hence, by the functoriality of the Clifford product, $\bigwedge^\bullet W$ and $\bigwedge^\bullet W'$ are subalgebras of $\mathrm{C}(\mathbb{C}^n, q)$. Consider the linear map

$$\begin{aligned} \textstyle\bigwedge^\bullet W \otimes \bigwedge^\bullet W' &\longrightarrow \mathrm{C}(\mathbb{C}^n, q); \\ a \otimes a' &\longmapsto aa' \end{aligned}$$

this is not a homomorphism of algebras. I claim that this is an isomorphism of $\mathbb{C}$-vector spaces.

Since both the domain and the codomain have dimension 2^n, it is enough to show that it is surjective. It is well know that the $\mathbb{C}$-linear map

$$\begin{aligned} \textstyle\bigwedge^\bullet W \otimes \bigwedge^\bullet W' &\longrightarrow \textstyle\bigwedge^\bullet (W \oplus W') \\ x \otimes x' &\longmapsto x \wedge x' \end{aligned}$$

is an isomorphism, and then surjectivity follows by standard arguments from Proposition 5.7.

For each subset $I \subseteq \{1, \dots, n\}$, write

$$\begin{aligned} w'_I &\stackrel{\text{def}}{=} w'_{i_1} \dots w'_{i_k} \\ &= w'_{i_1} \wedge \dots \wedge w'_{i_k}, \end{aligned}$$

where $I = \{i_1, \dots, i_k\}$, and $i_1 < \dots < i_k$. Of course $w'_\emptyset = 1$. We set $\omega' = w'_{\{1,\dots,n\}} = w'_1 \wedge \dots \wedge w'_n$. Consider the $\mathbb{C}$-linear map

$$\begin{aligned} \textstyle\bigwedge^\bullet W &\longrightarrow \mathrm{C}(\mathbb{C}^n, q); \\ x &\longmapsto x\omega' \end{aligned}$$

it is the composition of the embedding

$$\begin{aligned} \textstyle\bigwedge^\bullet W \otimes \bigwedge^\bullet W' &\longrightarrow \textstyle\bigwedge^\bullet W \otimes \bigwedge^\bullet W' \\ x &\longmapsto x \otimes \omega' \end{aligned}$$

with the isomorphism $\bigwedge^\bullet W \otimes \bigwedge^\bullet W' \simeq \mathrm{C}(\mathbb{C}^n, q)$ above, hence it is an embedding.

Proposition 8.3. *The image of $\bigwedge^\bullet W$ in $\mathrm{C}(\mathbb{C}^n, q)$ is the left ideal generated by ω'. Furthermore if $w \in W$, $w' \in W'$ and $x \in \bigwedge^\bullet W$, we have*

$$(w + w')x\omega' = (w \wedge x + 2w' \vdash x)\omega'.$$

So $\bigwedge^\bullet W$ becomes a left ideal in $\mathrm{C}(\mathbb{C}^n, q)$, and the induce left module structure is precisely that given by Theorem 8.2.

Remark 8.4. The bases $w_1, \dots, w_m$ and $w'_1, \dots, w'_m$ satisfy the condition $q(w_i, w'_i) = 1/2$. This seemingly strange factor $1/2$ has been introduced so that the element w'_i of the Clifford algebra acts on the spinor space $\bigwedge^\bullet W$ by sending w_i to 1. This is sometimes convenient in calculations.

Proof. The image of $\bigwedge^\bullet W$ is obviously contained in the left ideal generated by ω'. To prove the opposite inclusion, notice that for each $w' \in W'$ we have $w' \wedge \omega' = 0$ (for reasons of degree) and $w' \vdash \omega' = 0$ (because W' is totally isotropic); hence

$w'\omega' = 0$. Hence $w'_I\omega' = 0$ for any $I \subseteq \{1,\dots,n\}$ with $I \neq \emptyset$. Any element of $\mathrm{C}(V,q)$ can be written in the form $\sum_I x_I w'_I$ with $a_I \in \bigwedge^\bullet W$, hence every element of the left ideal generated by ω' is of the form

$$\sum_I x_I w'_I \omega' = x_\emptyset \omega'.$$

This proves the first part.

For the second part, notice that for any $w' \in W'$ and any $x \in \bigwedge^\bullet W$ we have

$$\begin{aligned} xw' &= x \wedge w' + x \dashv w' \\ &= (-1)^{|x|}(w' \wedge x - w' \vdash x) \\ &= (-1)^{|x|}(w'x - 2w' \vdash x); \end{aligned}$$

since $xw'\omega' = 0$, we get the formula $(w'x)\omega' = 2(w' \vdash x)\omega'$; hence

$$\begin{aligned} (w+w')x\omega' &= (wx)\omega' + (w'x)\omega' \\ &= (w \wedge x)\omega' + 2(w' \vdash x)\omega', \end{aligned}$$

as claimed. ♠

Proposition 8.5. *$\bigwedge^\bullet W$ is a simple module over $\mathrm{C}(\mathbb{C}^n, q)$.*

Proof. We will use the bases w_i and w'_j of W and W' constructed above, with the property that $q(w_i, w'_j)$ is 0 for $i \neq j$, and $1/2$ for $i = j$. For each $I \subseteq \{1,\dots,n\}$ consider the element w_I, defined as usual, in such a way that the w_I form a basis of $\bigwedge^\bullet W$. Then, because of the way the $\mathrm{C}(\mathbb{C}^n, q)$ module structure on $\bigwedge^\bullet W$ is defined, we have

$$w'_i w_I = \begin{cases} 0 & \text{if } i \notin I \\ \pm w_{I\setminus\{i\}} & \text{if } i \in I; \end{cases}$$

hence if I and J are subsets of $\{1,\dots,n\}$ we have that $w'_J w_I = 0$ whenever $|I| < |J|$, or $|I| = |J|$ and $I \neq J$; while $w'_J w_J = \pm 1$. Assume that $Z \subseteq \bigwedge^\bullet W$ is a non-zero submodule; choose $z \in Z \setminus \{0\}$, and write z as $\sum_I \alpha_I w_I$. Choose $J \subseteq \{1,\dots,n\}$ among those with $\alpha_J \neq 0$, such that $|J|$ is as large as possible. Then $w'_J z = \pm\alpha_J$; hence Z contains a non-zero scalar, hence it contains 1. Then it contains $w_I = w_I 1$ for all I, so $Z = \bigwedge^\bullet W$. ♠

This gives the structure of the Clifford algebra $\mathrm{C}(\mathbb{C}^n, q)$.

Theorem 8.6. *The homomorphism*

$$\mathrm{C}(\mathbb{C}^n, q) \longrightarrow \mathrm{End}_{\mathbb{C}}\bigl(\textstyle\bigwedge^\bullet W\bigr)$$

that comes from the $\mathrm{C}(\mathbb{C}^n, q)$-module structure on $\bigwedge^\bullet W$ is an isomorphism.

Proof. From Proposition 8.5 and Corollary A.14 we see that it is surjective. Since both spaces have dimension 2^n we are done. ♠

From this we also see the structure of the even part $\mathrm{C}^+(\mathbb{C}^n, q)$. The point is that $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ has a $\mathbb{Z}/2\mathbb{Z}$-grading, in which the even part is formed by endomorphisms of $\bigwedge^\bullet W$ preserving the parity of vectors in $\bigwedge^\bullet W$, hence it is a product $\mathrm{End}_{\mathbb{C}}(\bigwedge^+ W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^- W)$, while the odd part is formed by endomorphisms that send even vectors into odd vectors, and odd vectors into even vectors. The image of a vector in $\mathbb{C}^n$ into $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ is odd, as one sees readily from the condition in Theorem 8.2; from the uniqueness of the grading (Proposition 4.3) we get that the isomorphism is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Hence the module $\bigwedge^\bullet W$ splits as a direct sum of two $C^+(\mathbb{C}^n, q)$-submodules $\bigwedge^+ W$ and $\bigwedge^- W$, and we obtain the following.

Theorem 8.7. *The homomorphism*

$$C^+(\mathbb{C}^n, q) \longrightarrow \operatorname{End}_{\mathbb{C}}\big(\bigwedge^+ W\big) \times \operatorname{End}_{\mathbb{C}}\big(\bigwedge^- W\big)$$

that comes from the $C^+(\mathbb{C}^n, q)$-module structures on $\bigwedge^+ W$ and $\bigwedge^- W$ is an isomorphism.

In particular, the two $C^+(\mathbb{C}^n, q)$-modules $\bigwedge^+ W$ and $\bigwedge^- W$ are simple and not isomorphic.

Definition 8.8. The two $C^+(\mathbb{C}^n, q)$-modules $\bigwedge^+ W$ and $\bigwedge^- W$ are called *half-spinor* modules.

For later use, let us notice the following. If $f \in O_n(\mathbb{C})$, by the functorialy of the Clifford algebra f extends uniquely to an automorphism of $C^+(\mathbb{C}^n, q)$, that we still denote by f. For each $C^+(\mathbb{C}^n, q)$-module M, denote by M^f the module with the scalar multiplication twisted by f; that is, if $a \in C^+(\mathbb{C}^n, q)$ and $x \in M$, the new scalar product $a\dot{x}$ equals $f(a)x$. Since $\bigwedge^+ W$ and $\bigwedge^- W$ are the only simple modules over $C(\mathbb{C}^n, q)$, it is clear that there are two possibilities: either $\big(\bigwedge^+ W\big)^f \simeq \bigwedge^+ W$ and $\big(\bigwedge^- W\big)^f \simeq \bigwedge^- W$, or $\big(\bigwedge^+ W\big)^f \simeq \bigwedge^- W$ and $\big(\bigwedge^- W\big)^f \simeq \bigwedge^+ W$.

Proposition 8.9.

(a) *If $f \in SO_n(\mathbb{C})$, then $\big(\bigwedge^+ W\big)^f \simeq \bigwedge^+ W$ and $\big(\bigwedge^- W\big)^f \simeq \bigwedge^- W$.*

(b) *If $f \in O_n \setminus SO_n$, then $\big(\bigwedge^+ W\big)^f \simeq \bigwedge^- W$ and $\big(\bigwedge^- W\big)^f \simeq \bigwedge^+ W$.*

The proof is postponed to page 33, after the proof of Theorem 9.9.

The odd case. When $n = 2m + 1$ the situation is a little different: the spinor space $\bigwedge^\bullet W$ has *two* module structure over $C(\mathbb{C}^n, q)$. We have $\mathbb{C}^n = W \oplus W' \oplus U$, where $U = \langle u_0 \rangle$ as above; we let $W \oplus W'$ operate by the same formula as before; u_0 will act on $x \in \bigwedge^\bullet W$ by multiplication by $(-1)^{|x|}$ in one case, and by multiplication by $(-1)^{|x|+1}$ in the other.

Theorem 8.10. *There are two module structures on $\bigwedge^\bullet W$, uniquely determined by the formulas*

$$(w + w' + \lambda u_0)x = w \wedge x + 2w' \vdash x + (-1)^{|x|}\lambda x$$

and

$$(w + w' + \lambda u_0)x = w \wedge x + 2w' \vdash x - (-1)^{|x|}\lambda x$$

for any $w \in W$, $w' \in W'$ and $\lambda \in \mathbb{C}$.

Proof. As in the proof of Theorem 8.2, it is enough to show that the $\mathbb{C}$-linear map $\psi\colon C(\mathbb{C}^n, q) \to \operatorname{End}_{\mathbb{C}}\big(\bigwedge^\bullet W\big)$ defined by one of the two formulas

$$\psi(w + w' + \lambda u_0)x = w \wedge x + 2w' \vdash x \pm (-1)^{|x|}\lambda x$$

has the property that $\psi(w + w' + \lambda u_0)^2 = q(w + w') + \lambda^2$.

We have $q(w + w' + \lambda u_0) = 2q(w, w') + \lambda^2$, and

$$\begin{aligned}\psi(w + w' + \lambda u_0)^2 x &= \psi(w + w' + \lambda u_0)(w \wedge x + 2w' \vdash x \pm (-1)^{|x|}\lambda x)\\ &= 2w \wedge (w' \vdash x) \pm (-1)^{|x|}\lambda w \wedge x\\ &\quad + 2w' \vdash (w \wedge x) \pm (-1)^{|x|}\lambda w \wedge x\\ &\quad \pm (-1)^{|x|+1}\lambda w \wedge x \pm (-1)^{|x|-1}\lambda w' \vdash x + \lambda^2 x\\ &= 2w \wedge (w' \vdash x) + 2q(w, w')x - 2w \wedge (w' \vdash x) + \lambda^2 x\\ &= q(w + w' + \lambda u_0)x,\end{aligned}$$

as claimed. ♠

We denote the two module structures by $\bigwedge_1^\bullet W$ and $\bigwedge_2^\bullet W$ respectively.

As in the even case, these module structures can be seen as given by embedding the spinor space $\bigwedge^\bullet W$ into $C(\mathbb{C}^n, q)$ as left ideals.

First, $\bigwedge^\bullet W$, $\bigwedge^\bullet W'$ and $\bigwedge^\bullet U$ are all embedded in $C(\mathbb{C}^n, q)$ as subalgebras, because the restriction of q to W, W' and U are all trivial; and the $\mathbb{C}$-linear map

$$\begin{aligned}\textstyle\bigwedge^\bullet W \otimes \bigwedge^\bullet W' \otimes \bigwedge^\bullet U &\longrightarrow \textstyle\bigwedge^\bullet (W \oplus W' \oplus U)\\ a \otimes a' \otimes b &\longmapsto aa'b\end{aligned}$$

is an isomorphism. Set $\omega' = w'_1 \wedge \cdots \wedge w'_m$ as before. Since $u_0^2 = 1$ we have $u_0(1 \pm u_0) = \pm u_0$; furthermore $u_0\omega' = u_0 \wedge \omega' = (-1)^m \omega' \wedge u_0 = (-1)^m \omega' u_0$. We set

$$\begin{aligned}f_1 &= (1 + u_0)\omega'\\ &= \omega'\bigl(1 + (-1)^m u_0\bigr)\end{aligned}$$

and

$$\begin{aligned}f_2 &= (1 - u_0)\omega'\\ &= \omega'\bigl(1 - (-1)^m u_0\bigr).\end{aligned}$$

If $w' \in W'$ we have $w' f_i = 0$, while $u_0 f_1 = f_1$, $u_0 f_2 = -f_2$; hence the elements of $\bigwedge^\bullet W' \subseteq \bigwedge^\bullet V$ and $\bigwedge^\bullet U$ send f_i into a multiple of itself. We deduce that the embedding $\bigwedge^\bullet W \hookrightarrow \bigwedge^\bullet W \otimes \bigwedge^\bullet W' \bigwedge^\bullet U \simeq C(\mathbb{C}^n, q)$ that sends x into $x f_i$ identifies $\bigwedge^\bullet W$ with the left ideal generated by f_i.

Proposition 8.11. *If $w \in W$, $w' \in W'$, $\lambda \in \mathbb{C}$ and $x \in \bigwedge^\bullet W$, we have*

$$(w + w' + \lambda u_0) x f_1 = \bigl(w \wedge x + 2w' \vdash x + (-1)^{|x|}\lambda\bigr) x f_1$$

and

$$(w + w' + \lambda u_0) x f_2 = \bigl(w \wedge x + 2w' \vdash x - (-1)^{|x|}\lambda\bigr) x f_2$$

Hence the two module structures on these ideals are those whose existence is asserted in Theorem 8.10.

We leave the proof to the reader.

To investigate these modules, notice that the embedding $\mathbb{C}^{2m} = W \oplus W' \subseteq \mathbb{C}^n$ induces an embedding of algebras $C(\mathbb{C}^{2m}, q) \subseteq C(\mathbb{C}^n, q)$ (by abuse of notation we denote by q both the quadratic form on $\mathbb{C}^n$ and its restriction to $\mathbb{C}^{2m}$). The restrictions of $\bigwedge_1^\bullet W$ and of $\bigwedge_2^\bullet W$ to $C(\mathbb{C}^{2m}, q)$ both coincide with the spinor module on $C(\mathbb{C}^{2m}, q)$.

Proposition 8.12. *$\bigwedge_1^\bullet W$ and $\bigwedge_2^\bullet W$ are simple non-isomorphic modules over $\mathrm{C}(\mathbb{C}^n, q)$.*

Proof. The restrictions of $\bigwedge_1^\bullet W$ and of $\bigwedge_2^\bullet W$ to $\mathrm{C}(\mathbb{C}^{2m}, q)$ both coincide with the spinor module on $\mathrm{C}(\mathbb{C}^{2m}, q)$, which is simple, according to Proposition 8.5; this shows that $\bigwedge_1^\bullet W$ and $\bigwedge_2^\bullet W$ are simple.

Assume that $\phi\colon \bigwedge_1^\bullet W \simeq \bigwedge_2^\bullet W$ is an isomorphism of $\mathrm{C}(\mathbb{C}^n, q)$-modules. The element u_0 of $\mathrm{C}(\mathbb{C}^n, q)$ acts as the identity on $\bigwedge_1^+ W$ and on $\bigwedge_2^- W$, and as multiplication by -1 on $\bigwedge_1^- W$ and on $\bigwedge_2^+ W$; hence ϕ induces an isomorphism of $\bigwedge_1^+ W$ with $\bigwedge_2^- W$. The restrictions of $\bigwedge_1^+ W$ and $\bigwedge_2^- W$ to the even part $\mathrm{C}^+(\mathbb{C}^{2m}, 1)$ coincide with $\bigwedge^+ W$ with $\bigwedge^- W$, and ϕ induces an isomorphism of $\mathrm{C}^+(\mathbb{C}^{2m}, 1)$-modules $\bigwedge^+ W \simeq \bigwedge^- W$; but, according to Theorem 8.7, this isomorphism can not exist. ♠

Theorem 8.13. *The homomorphism*

$$\mathrm{C}(\mathbb{C}^n, q) \longrightarrow \mathrm{End}_{\mathbb{C}}\big(\textstyle\bigwedge^\bullet W\big) \times \mathrm{End}_{\mathbb{C}}\big(\textstyle\bigwedge^\bullet W\big)$$

that comes from the two $\mathrm{C}(\mathbb{C}^n, q)$-modules structure on $\bigwedge^\bullet W$ is an isomorphism.

Proof. From Proposition 8.12 and Corollary A.14 we see that it is surjective. Since both spaces have dimension 2^n we are done. ♠

We will also need to describe the even part $\mathrm{C}^+(\mathbb{C}^n, q)$. For this, we need to produce a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{C}(\mathbb{C}^n, q) = \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$, such that $V \subseteq \mathrm{C}(\mathbb{C}^n, q)$ be composed by elements of degree 1. This can be done by explotting the natural $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$, used in the even case, such that the even elements are those sending $\bigwedge^+ W$ into $\bigwedge^+ W$ and $\bigwedge^- W$ into $\bigwedge^- W$, while an odd element sends $\bigwedge^+ W$ into $\bigwedge^- W$ and $\bigwedge^- W$ into $\bigwedge^+ W$. In this $\mathbb{Z}/2\mathbb{Z}$-grading, the even elements of $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ consisting of pairs (f, g), such that $g^+ = f^+$ and $g^- = -f^-$ (here the superscript $^+$ and $^-$ denote even part and odd part), while the odd part is made of elements with $g^+ = -f^+$ and $g^- = f^-$. We leave it to the reader to check that this is a grading; while it follows immediately from the definition of the action of V on $\bigwedge^\bullet W$ that all the elements of $V \subseteq \mathrm{C}(\mathbb{C}^n, q) = \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ are odd.

Thus $\mathrm{C}^+(\mathbb{C}^n, q)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ consisting of elements of the form $\big(f, \epsilon(f)\big)$, where ϵ is the main involution of $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$, which changes the sign of the odd elements. From this we see that the first projection $\mathrm{C}^+(\mathbb{C}^n, q) \to \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ is an isomorphism, and identifying $\mathrm{C}(\mathbb{C}^n, q)$ with $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$, the second projection is the main involution ϵ. The two modules $\bigwedge_1^\bullet W$ and $\bigwedge_2^\bullet W$ become isomorphic, and simple, when restricted to $\mathrm{C}^+(\mathbb{C}^n, q)$, the isomorphism being the main involution of $\bigwedge^\bullet W$. Let us record this as follows.

Theorem 8.14. *The homomorphism $\mathrm{C}^+(\mathbb{C}^n, q) \to \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ coming from either of the two $\mathrm{C}(\mathbb{C}^n, 1)$ module structures on $\bigwedge^\bullet W$ is an isomorphism.*

The canonical pairings. The spinor space $\bigwedge^\bullet W$ carries two very important bilinear form. Fix an element $\omega \in \bigwedge^m W \setminus \{0\}$; for example, we can take $\omega \overset{\text{def}}{=} w_1 \wedge \cdots \wedge w_m$. We define a linear function $\int\colon \bigwedge^\bullet W \to \mathbb{C}$ by the formula

$$x_m = \Big(\int x\Big)\omega$$

where $x \in \bigwedge^\bullet W$, and x_m denotes the component of degree m of x.

Definition 8.15. The *canonical pairings* $\beta\colon \bigwedge^\bullet W \times \bigwedge^\bullet W \to \mathbb{C}$ and $\overline{\beta}\colon \bigwedge^\bullet W \times \bigwedge^\bullet W \to \mathbb{C}$ are the bilinear forms defined by the formulas

$$\beta(x, y) = \int x^{\mathrm{t}} \wedge y$$

and

$$\overline{\beta}(x, y) = \int \overline{x} \wedge y.$$

Since $\overline{x} = \epsilon(x)^{\mathrm{t}}$, the two pairings are related by the formula

$$\overline{\beta}(x, y) = \beta\bigl(\epsilon(x), y\bigr).$$

Proposition 8.16. *The pairings β and $\overline{\beta}$ are non-degenerate.*

The pairing β is symmetric when $m \equiv 0$ or $m \equiv 1 \pmod 4$, and alternating when $m \equiv 2$ or $m \equiv 3 \pmod 4$.

The pairing $\overline{\beta}$ is symmetric when $m \equiv 0$ or $m \equiv 3 \pmod 4$, and alternating when $m \equiv 2$ or $m \equiv 1 \pmod 4$.

Proof. The fact that β is non-degenerate is clear by the following easy fact. If $\{w_I\}$ is the usual basis of $\bigwedge^\bullet W$, where I ranges over the subsets of $\{1, \dots, n\}$, then $\beta(w_I, w_J) = 0$ unless I and J are complementary, in which case $\beta(w_I, w_J) = \pm 1$. Then it follows that $\overline{\beta}$ is also non-degenerate, because it differs by in an automorphism of the first factor.

For the second part of the statement, we need to prove the formula $\beta(y, x) = (-1)^{m(m-1)/2}\beta(x, y)$ for all x and y in $\bigwedge^\bullet W$. We may assume that x and y are homogeneous; and then both sides are 0 unless $|x| + |y| = m$, so we assume this. We have

$$\begin{aligned} \beta(y, x) &= \int y^{\mathrm{t}} \wedge x \\ &= \int (x^{\mathrm{t}} \wedge y)^{\mathrm{t}} \\ &= \int (-1)^{\frac{m(m-1)}{2}} x^{\mathrm{t}} \wedge y \\ &= (-1)^{\frac{m(m-1)}{2}} \beta(x, y) \end{aligned}$$

as claimed.

The proof of the third statement is similar, using the formula $\overline{x} = (-1)^{\frac{|x|(|x|+1)}{2}} x$. ♠

We have seen that $\mathrm{C}(\mathbb{C}^n, q)$ is isomorphic to $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ when n is even, and to $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ when n is odd. The canonical pairing allows to interpret the transposition on the Clifford algebra directly in terms of algebras of endomorphisms. The non-degenerate pairing β allows to define a transposition $f \mapsto f^{\mathrm{t}}$ in the algebra $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ by the formula

$$\beta\bigl(f^{\mathrm{t}}(x), y\bigr) = \beta\bigl(x, f(y)\bigr)$$

for all endomorphisms $f\colon \bigwedge^\bullet W \to \bigwedge^\bullet W$ and all x and y in $\bigwedge^\bullet W$. The function from $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ to itself that sends f into f^{t} is an anti-automorphism, that is, it is $\mathbb{C}$-linear, and such that $(fg)^{\mathrm{t}} = g^{\mathrm{t}} f^{\mathrm{t}}$ for any f and g in $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$.

Notice that by choosing a different element $\omega \in \bigwedge^m W \setminus \{0\}$ we change β by a scalar, and this does not change the transposition constructed above.

Proposition 8.17. *Suppose that $n = 2m$. The main anti-automorphism $-^{\mathrm{t}}$ in the Clifford algebra $\mathrm{C}(\mathbb{C}^n, q)$ corresponds to the transposition in $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ relative to the pairing β.*

Proof. Since $\mathbb{C}^n \subseteq \mathrm{C}(\mathbb{C}^n, q) \simeq \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ generates $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ as an algebra, it is enough to prove that the transposition relative to β leaves the elements of $\mathbb{C}^n$ invariant. Since $\mathbb{C}^n = W \oplus W'$, and because of the way the action of $\mathbb{C}^n$ is defined, it is enough to prove that $\beta(w \wedge x, y) = \beta(x, w \wedge y)$ and $\beta(w' \vdash x, y) = \beta(x, w' \vdash y)$ for any $w \in W$, $w' \in W'$, x and $y \in \bigwedge^\bullet W$.

The first equality is completely straightforward:

$$\begin{aligned}\beta(w \wedge x, y) &= \int (w \wedge x)^{\mathrm{t}} \wedge y \\ &= \int x^{\mathrm{t}} \wedge w \wedge y \\ &= \beta(x, w \wedge y).\end{aligned}$$

For the second one, we may assume that x and y are homogeneous. Furthermore, both sides of the inequality are 0, unless $|x| + |y| = m + 1$, so we may assume that $|x| + |y| = m + 1$. Then $x^{\mathrm{t}} \wedge y = 0$, so

$$\begin{aligned}0 &= w' \vdash (x^{\mathrm{t}} \wedge y) \\ &= (w' \vdash x^{\mathrm{t}}) \wedge y + (-1)^{|x|} x^{\mathrm{t}} \wedge (w' \vdash y) \\ &= (x \dashv w')^{\mathrm{t}} \wedge y + (-1)^{|x|} x^{\mathrm{t}} \wedge (w' \vdash y) \\ &= (-1)^{|x|+1} \big((w' \vdash x)^{\mathrm{t}} \wedge y - x^{\mathrm{t}} \wedge (w' \vdash y) \big);\end{aligned}$$

by applying $\int$ to the last line we get

$$\begin{aligned}0 &= \int (w' \vdash x)^{\mathrm{t}} \wedge y - \int x^{\mathrm{t}} \wedge (w' \vdash y) \\ &= \beta(w' \vdash x, y) - \beta(x, w' \vdash y);\end{aligned}$$

and this concludes the proof. ♠

The odd case is a little more subtle. In this case $\mathrm{C}(\mathbb{C}^n, q)$ is isomorphic to $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$; each of the two factors has a transposition $-^{\mathrm{t}}$ relative to β.

Proposition 8.18. *Suppose that $n = 2m + 1$. Then the anti-automorphism of the algebra $\mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$ that corresponds to the main anti-automorphism of $\mathrm{C}(\mathbb{C}^n, q)$ is given by the formula*

$$(f, g) \longmapsto (f^{\mathrm{t}}, g^{\mathrm{t}})$$

when m is even, and by

$$(f, g) \longmapsto (g^{\mathrm{t}}, f^{\mathrm{t}})$$

when m is odd.

Proof. We need to check that the image of an element of $\mathbb{C}^n = W \oplus W' \oplus U$ in invariant under the anti-automorphism above. We have done this in the proof of Proposition 8.18 for the elements of W and W': consider now the action of the generator u_0 of U. If ϵ denotes the main involution of $\bigwedge^\bullet W$, defined by the formula

$\epsilon(x) = (-1)^{|x|}x$, then u_0 acts via $(\epsilon, -\epsilon)$. I claim that $\epsilon^{\mathrm{t}} = (-1)^m \epsilon$. This is equivalent to the equality $\beta\bigl(\epsilon(x), y\bigr) = (-1)^m \beta\bigl(x, \epsilon(y)\bigr)$; to check this we may assume that x ane y are homogeneous and $|x| + |y| = m$. Then we have

$$\beta\bigl(\epsilon(x), y\bigr) = (-1)^{|x|}\beta(x, y)$$

while

$$\begin{aligned}\beta\bigl(x, \epsilon(y)\bigr) &= (-1)^{|y|}\beta(x, y)\\ &= (-1)^{m-|x|}\beta(x, y)\\ &= (-1)^m(-1)^{|x|}\beta(x, y)\end{aligned}$$

from which the equality follows.

Thus, if m is even $\epsilon^{\mathrm{t}} = \epsilon$, and $(\epsilon, -\epsilon)$ is invariant under the involution $(f, g) \mapsto (f^{\mathrm{t}}, g^{\mathrm{t}})$; while if m is odd $\epsilon^{\mathrm{t}} = -\epsilon$, and $(\epsilon, -\epsilon)$ is invariant under the involution $(f, g) \mapsto (g^{\mathrm{t}}, f^{\mathrm{t}})$. This concludes the proof. ♠

9. Pin and Spin groups

In this section K will be either $\mathbb{R}$ or $\mathbb{C}$; V will correspondingly denote either $\mathbb{R}^n$ or $\mathbb{C}^n$. We will assume that n is at least 1.

In both cases, the quadratic form will be given by the formula $q(x) = -|a|^2$; and the notations O_n and SO_n will refer to either $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$ or $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$.

Definition 9.1. The *pin group* Pin_n is the set of elements $a \in \mathrm{C}(V, q)$ satisfying the following conditions.

(a) $a \in \mathrm{C}(V, q)$ is either even or odd.
(b) $a\overline{a} = 1$.
(c) For any $v \in V$, $ava^{-1} = av\overline{a}$ is in V.

The *spin group* Spin_n is the set of even elements in Pin_n

Remark 9.2. The terminology "pin group" is a joke attributed to J.P. Serre. The term"spin group" comes from the "spin representation" that Dirac introduced to analyze the phenomenon of spin of elementary particles; but Serre suggested that it might mean "special pin group", like SO_n is a "special orthogonal group" and SL_n is a "special linear group".

We will write $\mathrm{Pin}_n(\mathbb{R})$, $\mathrm{Pin}_n(\mathbb{C})$, $\mathrm{Spin}_n(\mathbb{R})$ and $\mathrm{Spin}_n(\mathbb{C})$ when we need to distinguish between the real and complex case.

Clearly, Pin_n and Spin_n are subgroups of the group of units in $\mathrm{C}(V, q)$.

For each $a \in \mathrm{Pin}_n$, define a linear map $\rho(a)\colon V \to V$ by the formula

$$v \longmapsto \epsilon(a)v\overline{a}.$$

I claim that $\rho(a)$ is orthogonal: in fact, we have

$$\begin{aligned}|\rho(a)v|^2 &= -q(\rho(a)v)\\ &= -(\rho(a)v)^2\\ &= -av\bar{a}av\bar{a}\\ &= -avv\bar{a}\\ &= -q(v)a\bar{a}\\ &= |v|^2\end{aligned}$$

for any $v \in V$. This defines a map $\rho\colon \mathrm{Pin}_n \to \mathrm{O}_n$; it is easy to see that this is a homomorphism of groups.

Remark 9.3. The embedding $\mathrm{C}(\mathbb{R}^n, q) \subseteq \mathrm{C}(\mathbb{C}^n, q)$ sends $\mathrm{Pin}_n(\mathbb{R})$ into $\mathrm{Pin}_n(\mathbb{C})$ and $\mathrm{Spin}_n(\mathbb{R})$ into $\mathrm{Spin}_n(\mathbb{C})$.

Remark 9.4. When $a \in V$, we have that $a\bar{a} = -a^2 = -q(a) = |a|^2$; furthermore

$$\begin{aligned}av\bar{a} &= -ava\\ &= -a\big(v \wedge a + q(a,v)\big)\\ &= -a \wedge \big(v \wedge a + q(a,v)\big) - a \vdash \big(v \wedge a + q(a,v)\big)\\ &= -a \wedge v \wedge a - q(a,v)a - q(a,v)a + q(a)v\\ &= q(a)v - 2q(a,v)a \in V.\end{aligned}$$

for any $v \in V$. Hence Pin_n contains all vectors of length 1 in V.

If $a \in V$, $|a|^2 = 1$, then

$$\begin{aligned}\rho(a)v &= -av\bar{a}\\ &= v - 2\langle a \mid v\rangle a.\end{aligned}$$

Hence $\rho(a)a = -a$ and $\rho(a)v = v$ whenever v is orthogonal to a: in other words, $\rho(a)$ is the reflexion with respect to the hyperplane orthogonal to a.

Example 9.5. Consider the case $n = 1$. Then $\mathrm{C}(V,q)$ is a 2-dimensional algebra over K, with basis 1, e_1, and $e_1^2 = -1$. Conjugation is given by the formula $\overline{a_0 + a_1e_1} = a_0 - a_1e_1$. All the elements of the algebra satisfy the condition that $ava^{-1} \in V$ for all $v \in V$, since the algebra is commutative. The even element satisfying $a\bar{a} = 1$ are ± 1, while the odd elements are $\pm e_1$. Since $e_1^2 = -1$, the group Pin_1 is cyclic of order 4, generated by e_1, while Spin_1 is the subgroup $\{\pm 1\}$.

The group O_1 is cyclic of order 2, generated by the reflexion $e_1 \mapsto -e_1$; by Remark 9.4, or by an immediate computation, we see that $\rho\colon \mathrm{Pin}_1 \to \mathrm{O}_1$ sends e_1 into this reflexion. The kernel of this homomorphism is $\{\pm 1\} = \mathrm{Spin}_1$.

In this case the real and complex groups coincide.

Example 9.6. Let us analyze the case $n = 2$. The group Spin_2 is contained in the group of units of the algebra $\mathrm{C}^+(V,q)$, which is 2-dimensional, with basis 1, $\iota \overset{\mathrm{def}}{=} e_1e_2$. The product is determined by the condition $\iota^2 = -1$, and conjugation is given by the formula $\overline{a + b\iota} = a - b\iota$; furthermore

$$(a + b\iota)\overline{(a + b\iota)} = a^2 + b^2.$$

Also, we have that the odd part of $C(V,q)$ in this case coincides with V; so, since $av\bar{a}$ is odd for any $v \in V$ and any $a \in C^{+}(V,q)$, we have that it is automatically in V. Hence Spin_2 is given by the elements $a + b\iota$ satisfying the condition $a^2 + b^2 = 1$. So $\mathrm{Spin}_2(\mathbb{R})$ is the circle group $\mathbb{S}^1$. In the complex case we have the factorization

$$a^2 + b^2 = (a + \mathrm{i}b)(a + \mathrm{i}b)$$

$\mathrm{Spin}_2(\mathbb{C})$ is isomorphic to $\mathbb{C}^*$, the isomorphism $\mathrm{Spin}_2(\mathbb{C}) \to \mathbb{C}^*$ being the map $a + b\iota \mapsto a + \mathrm{i}b$.

Notice that Spin_2 is isomorphic to SO_2; the isomorphism ϕ is obtained by sending an element $a + b\iota$ into the matrix

$$\phi(a + b\iota) \stackrel{\text{def}}{=} \begin{pmatrix} a & -b \\ b & a \end{pmatrix};$$

the existence of this isomorphism a peculiarity of the case $n = 2$. We have the relations $\iota e_1 = e_2 = -e_1\iota$ and $\iota e_2 = -e_1 = -e_2\iota$, from which we deduce that $\phi(a)v = av$ for all $a \in \mathrm{Spin}_2$ and $v \in V$.

However, the homomorphism ρ is not an isomorphism. In fact, we have that ι anticommutes with all the elements of V, hence $v\bar{a} = av$ for all $a \in \mathrm{Spin}_2$ and $v \in V$; so $av\bar{a} = a^2v = \phi(a^2)v$. The homomorphism ρ can be identified with the map $a \mapsto a^2$ from $\mathbb{S}^1$ to itself (in the real case) or $\mathbb{C}^*$ to itself (in the complex case). This homomorphism is surjective, with kernel $\{\pm 1\}$.

The group Pin_2 is more complicated, we will describe it briefly without proofs. It contains Spin_2 as a subgroup of index 2; besides the elements Spin_2, it contains all the vectors of V of length 1. For any such vector v and any $a \in \mathrm{Spin}_2$, we have $vav^{-1} = -a$. Thus, it looks very much like O_2; but O_2 is a semidirect product of a cyclic group of order 2 with SO_2, while Pin_2 is not a semidirect product. One can show that there are exactly two extensions of a cyclic group of order 2 by $\mathrm{Spin}_2 = \mathrm{SO}_2$ inducing the given action of the cyclic group on Spin_2, and these are precisely O_2 and Pin_2.

Remark 9.7. The previous analysis of the structure of Spin_2 is very important. It can be applied in the following situation. Suppose that Z is a 2-dimensional subspace of V, such that the restriction of q to Z is non-degenerate (of course this is always verified when $K = \mathbb{R}$). Then Z has an orthogonal basis ϵ_1, ϵ_2 with $|\epsilon_1|^2 = |\epsilon_2|^2 = 1$; the Clifford algebra $C(Z, q|_Z)$ is isomorphic to $C(K^2, q)$; hence the elements of type $a + b\epsilon_1\epsilon_2$, with $a, b \in K$, $a^2 + b^2 = 1$, form a subgroup of the group of units of $C(Z, q|_Z)$ that is isomorphic to $\mathbb{S}^1$ or to $\mathbb{C}^*$, according to whether K is $\mathbb{R}$ or $\mathbb{C}$.

I claim that this subgroup is contained in Spin_n. In fact, the element $a + b\epsilon_1\epsilon_2$ can be written as $\epsilon_1(-a + b\epsilon_2)$; both ϵ_1 and $-a + b\epsilon_2$ are vectors of length 1, so are in Pin_n, so their product is an even element in Pin_n.

Thus, Spin_n contains many copies of $\mathbb{S}^1$ (in the real case) or of $\mathbb{C}^*$ (in the complex case).

When $K = \mathbb{C}$ it may happen that $q|_Z$ is degenerate. Suppose that is has rank 1; choose an element $v \in Z$ with $|v|^2 = 1$, and another element $u \in Z$ that generates the radical of $q|_Z$. For any $\lambda \in \mathbb{C}$, the element $1 + \lambda vu \in C(Z, q|_Z) \subseteq C(\mathbb{C}^n, q)$ is contained in $\mathrm{Spin}_n(\mathbb{C})$. In fact, it is clearly even, and it can be written as $v(-v + \lambda u)$: both factors v and $-v + \lambda u$ are vectors of length 1 in $\mathbb{C}^n$.

There is an embedding of algebraic varieties $\mathbb{C} \hookrightarrow \mathrm{Spin}_n(\mathbb{C})$. I claim that this is a homomorphism of groups; hence, it makes $\mathbb{C}$ into an algebraic subgroup of Spin_n. In fact, we have

$$\begin{aligned}(1+\lambda vu)(1+\lambda' vu) &= 1+(\lambda+\lambda')vu+\lambda\lambda'(vu)^2\\ &= 1+(\lambda+\lambda')vu-\lambda\lambda' v^2u^2\\ &= 1+(\lambda+\lambda')vu.\end{aligned}$$

Thus, $\mathrm{Spin}_n(\mathbb{C})$ contains also copies of the additive group $\mathbb{C}$.

Example 9.8. The previous examples were very anomalous; the case $n = 3$ is more representative of the general case. The even part $\mathrm{C}^+(V,q)$ is 4-dimensional, with a basis consisting of 1, $\epsilon_1 \overset{\text{def}}{=} e_2e_3$, $\epsilon_2 \overset{\text{def}}{=} e_3e_1$ and $\epsilon_3 \overset{\text{def}}{=} e_1e_2$. These elements satisfy the relations $\epsilon_i^2 = -1$, and $\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 0$ for all $i \neq j$; hence $\mathrm{C}^+(\mathbb{R}^3, q)$ is the quaternion algebra $\mathbb{H}$; while in the complex case we know that $\mathrm{C}^+(\mathbb{C}^3, q)$ is isomorphic to $\mathrm{M}_2(\mathbb{C})$. An explicit isomorphism can be obtained by setting

$$\epsilon_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i}\end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \quad \text{and} \quad \epsilon_3 = \epsilon_2\epsilon_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0\end{pmatrix}$$

Both in the real and complex case, conjugation is given by the formula

$$\overline{a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3} = a_0 - a_1\epsilon_1 - a_2\epsilon_2 - a_3\epsilon_3;$$

hence

$$(a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3)\overline{(a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3)} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Notice that for any $a \in \mathrm{C}^+(V,q)$ and $v \in V$, the element $av\bar{a}$ is odd, hence it is a combination of elements of degree 1 (that is, vectors in V) and 3. However, we have $\overline{av\bar{a}} = a\bar{v}\bar{a} = -av\bar{a}$; because of the formula $\bar{x} = (-1)^{|x|(|x|+1)/2}$, we see that the part of degree 3 in that $av\bar{a}$ is 0, hence $av\bar{a}$ is always in V. So, Spin_3 is the group of elements of $r\mathrm{C}^+(V,q)$ of the form $a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3$, with $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$.

When $K = \mathbb{R}$, we see that $\mathrm{Spin}_3(\mathbb{R})$ is the group $\mathbb{S}^3$ of quaternions of norm 1. In the complex case, we have identified the element $a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3$ with the matrix

$$\begin{pmatrix} a_0 + \mathrm{i}a_1 & a_2 + \mathrm{i}a_3 \\ -a_2 + \mathrm{i}a_3 & a_0 - \mathrm{i}a_1\end{pmatrix}$$

whose determinant is precisely $a_0^2 + a_1^2 + a_2^2 + a_3^2$. Hence, $\mathrm{Spin}_3(\mathbb{C})$ is isomorphic to $\mathrm{SL}_2(\mathbb{C})$.

Theorem 9.9.

(a) *The group* $\mathrm{Pin}_n(\mathbb{R})$ *is a compact Lie subgroup of* $\mathrm{C}(\mathbb{R}^n,q)^0$ *of dimension* $n(n-1)/2$. *The group* $\mathrm{Pin}_n(\mathbb{C})$ *is an algebraic subgroup of* $\mathrm{C}(\mathbb{C}^n,q)^0$ *of complex dimension* $n(n-1)/2$.

(b) *The map* $\rho\colon \mathrm{Pin}_n \to \mathrm{O}_n$ *is a homomorphism of Lie groups (when* $K = \mathbb{R}$*) or of algebraic groups (when* $K = \mathbb{C}$*). It is surjective, and its kernel is* $\{\pm 1\}$.

(c) *Every element of* Pin_n *is a product of vectors of length* 1 *in* V. *In particular,* Pin_n *is generated by the vectors* $v \in V$ *with* $|v| = 1$.

Proof. One checks immediately that Pin_n is a subgroup of the group of units $\mathrm{C}(V,q)^0$ in $\mathrm{C}(V,q)$. When $K=\mathbb{R}$, the group $\mathrm{C}(\mathbb{R}^n,q)^0$ is an open subset of the euclidean space $\mathrm{C}(V,q)=\mathbb{R}^{2^n}$; when $K=\mathbb{C}$, $\mathrm{C}(\mathbb{C}^n,q)^0$ will be a Zariski open subset of $\mathrm{C}(\mathbb{C}^n,q)=\mathbb{C}^{2^n}$. Multiplication is defined by a bilinear function: hence $\mathrm{C}(\mathbb{R}^n,q)^0$ becomes a Lie group, while $\mathrm{C}(\mathbb{C}^n,q)^0$ is an algebraic group.

It follows immediately from the definition that $\mathrm{Pin}_n(\mathbb{C})$ is a Zariski-closed subgroup subset of $\mathrm{C}(\mathbb{C}^n,q)$; so it is a Zariski-closed subgroup of $\mathrm{C}(\mathbb{C}^n,q)^0$, hence it is an algebraic group. On the other hand, $\mathrm{Pin}_n(\mathbb{R})$ is a closed subgroup of the Lie group $\mathrm{C}(\mathbb{R}^n,q)^0$, so it is a Lie subgroup ([Var84, Theorem 2.12.6]).

Notice that Spin_n and $\mathrm{Pin}_n\setminus\mathrm{Spin}_n$ are open and closed subsets of Pin_n, since they are obtained by intersecting Pin_n with the linear subspaces $\mathrm{C}^+(V,q)$ and $\mathrm{C}^-(V,q)$. The homomorphism ρ is differentiable (when $K=\mathbb{R}$) or algebraic (when $K=\mathbb{C}$), because it is defined by polynomial functions in each of these two open subsets.

To show that $\rho\colon \mathrm{Pin}_n\to\mathrm{O}_n$ is surjective, take a vector $a\in V$ with $|a|^2=1$, and consider the homomorphism $\rho(a)\in\mathrm{O}_n$; according to Remark 9.4, this a reflexion along the hyperplane orthogonal to a. Then the surjectivity of ρ follows from the following Lemma.

Lemma 9.10. *Every element of O_n is a product of reflexions along hyperplanes orthogonal to vectors of length* 1*.*

Sketch of proof. We proceed by induction on n, starting from the case $n=1$, which is obvious.

Let $A\in\mathrm{O}_n$. Set $v=e_n$, $w=Ae_n$. I claim that there exists $B\in\mathrm{O}_n$ that is product of at most 2 reflexions such that $Bv=w$.

Suppose that $|v-w|^2\neq 0$. Then one checks that the reflexion along the hyperplane orthogonal to $v-w$ switches v and w (the easiest way to do this is to notice that since $|v|^2=|w|^2=1$, $v-w$ and $v+w$ are orthogonal, hence this reflexion sends $v-w$ into $w-v$ and leave $v+w$ fixed).

If $|v-w|^2=0$, then $|v+w|^2\neq 0$, because $|v-w|^2+|v+w|^2=4$; then we can first reflect along the hyperplane orthogonal to w, sending w in $-w$, then reflect along the hyperplane orthogonal to $v+w$.

So we have $B^{-1}Ae_n=e_n$; hence $B^{-1}A\in\mathrm{O}_{n-1}\subseteq\mathrm{O}_n$, and we use the induction hypothesis. ♠

Obviously $\{\pm 1\}$ is contained in the kernel of $\rho\colon \mathrm{Pin}_n\to\mathrm{O}_n$, so we need to prove the reverse inclusion. Let $a\in\mathrm{Pin}_n$ be an element of the kernel of ρ. If a is odd, then $ava^{-1}=-av\bar{a}=-v$ for any $v\in V$; this mean that a anticommutes with v. On the other hand, if a is even then $ava^{-1}=av\bar{a}=v$ for all $v\in V$, and a commutes with v for all $v\in V$. Then the statement follows easily from the following lemma.

Lemma 9.11.

(a) *If an odd element of $\mathrm{C}(V,q)$ anticommutes with all the elements of V, then it is* 0*.*
(b) *If an even element of $\mathrm{C}(V,q)$ commutes with all the elements of V, then it is a scalar.*

Proof. For (a), suppose that a is odd, and $av+va=0$ for all $v\in V$. Since a contains only elements of odd degree, we have $v\wedge a=-a\wedge v$ and $v\vdash a=a\dashv v$, hence

$2v \vdash a = av + va = 0$ for all $v \in V$. By Lemma 3.4, a is a scalar, hence $a = 0$, which is a contradiction, because a is invertible.

For (b), suppose that a is even and $av - va = 0$ for all $v \in V$, and we have $v \wedge a = a \wedge v$ and $v \vdash a = -a \dashv v$. So $2v \vdash a = av - va = 0$ for all $v \in V$, and, again from Lemma 3.4, we conclude that a is a scalar. ♠

So a can not be odd, because a is invertible, hence not 0. So a is a scalar; since $a^2 = a\bar{a} = 1$ we have $a = \pm 1$, as we were claiming.

Since $\rho\colon \mathrm{Spin}_n \to \mathrm{O}_n$ is a surjective homomorphism of Lie groups, or algebraic groups, with kernel $\{\pm 1\}$, it is a double cover. So the dimension of Pin_n equals the dimension of O_n, which is $n(n-1)/2$. Furthermore, $\mathrm{Pin}_n(\mathbb{R})$ is compact, since $\mathrm{O}_n(\mathbb{R})$ is compact.

We have left to show that every element of Pin_n is a product of vectors of length 1. By Remark 9.4 and Lemma 9.10, and because the kernel of ρ is $\{\pm 1\}$, which is contained in the center of Pin_n, we have that every element of Pin_n can be written in the form $\pm v_1 \dots v_k$ for certain $v_1, \dots, v_k$ of length 1 in V. If the sign is negative, we use the fact that $e_1^2 = -1$ (recall that we are assuming $n \geq 1$) to write it as $e_1^2 v_1 \dots v_k$. This concludes the proof of the theorem. ♠

Proof of Proposition 8.9. Lift $f \in \mathrm{O}_n$ to an element $u \in \mathrm{Pin}_n \subseteq \mathrm{C}^+(\mathbb{C}^n, q)$; if $f \in \mathrm{SO}_n$ then u is even, otherwise it is odd. The extension of f to $\mathrm{C}^+(\mathbb{C}^n, q)$ is given by $a \mapsto uau^{-1}$; it is easy to check that the linear automorphism of $\bigwedge^\bullet W$ defined by $x \mapsto ux$ gives an isomorphism $\bigwedge^\bullet W \simeq \left(\bigwedge^\bullet W\right)^f$. On the other hand when u is even then this automorphism sends $\bigwedge^+ W$ and $\bigwedge^- W$ into themselves, while when it is odd it switches them. Thus the restrictions of this automorphism to $\bigwedge^+ W$ and to $\bigwedge^- W$ yield the desidered isomorphisms. ♠

Theorem 9.12.

(a) Spin_n *is a subgroup of index* 2 *in* Pin_n
(b) $\mathrm{Spin}_n = \rho^{-1}(\mathrm{SO}_n)$*; hence* Spin_n *is an open and closed subgroup of* Pin_n*.*
(c) $\rho\colon \mathrm{Spin}_n \to \mathrm{SO}_n$ *is surjective, with kernel* $\{\pm 1\}$*.*
(d) *If* $n \geq 2$*, then* Spin_n *is the connected component of the identity in* Pin_n*.*
(e) *If* $n \geq 2$*, the commutator subgroup of* Pin_n *is* Spin_n*.*
(f) *If* $n \geq 3$*, the commutator subgroup of* Spin_n *is* Spin_n *itself.*

Proof. Consider the function $\mathrm{Pin}_n \to \{\pm 1\}$ that sends each element of Pin_n into its sign. The group Pin_n contains at least one odd element, for example e_1, thus the homomorphism is surjective, and its kernel is precisely Spin_n. This proves (a).

For (b), take an element a of Pin_n, and write it as a product $v_1 \dots v_k$ of vectors of length 1. If k is even then a is also even, so it is in Spin_n, and $\rho(a)$ is a product of an even number of reflexions, hence it has determinant 1; while if k is odd a is not in Spin_n, and $\rho(a)$ has determinant -1.

(c) follows immediately from (b) and from Theorem 9.9 (b).

For (d), since the connected component of the identitity is a subgroup of Spin_n, and every element of Spin_n is the product of an even number of vectors of length 1, it is enough to show that every product vw of two vectors of length 1 is in this connected component.

If v and w are linearly dependend, then $w = \pm v$, and correspondingly $vw = \mp 1$; so we need to show that -1 is in the connected component of 1. It follows from

Remark 9.7 that the elements of the form $a + be_1e_2$, with a, $b \in K$ and $a^2 + b^2 = 1$ form a subgroup of Spin_n that is isomorphic and homeomorphic to the circle group ($\mathbb{S}^1$ or $\mathbb{C}^*$, depending on the base field), and so is connected; obviously this contains both 1 and -1.

Otherwise, let Z be the subspace generated by v and w. The rank of $q|_Z$ can be either 1 or 2. In case it has rank 1 (which can only happen when K is $\mathbb{C}$), let u be a generator of the radical of $q|_Z$. The vectors of length 1 in Z all have the form $\pm v + \alpha u$, with $\alpha \in \mathbb{C}$. If $w = -v + \alpha u$, then $vw = 1 + \lambda vu$, so vw is in the image of the map $\mathbb{C} \to \mathrm{Spin}_n$ that sends λ into $1 + \lambda vu$, thus it is in the connected component of 1. If w has the form $v + \lambda u$, then $-w$ has the required form, and so $-vw = v(-w)$ is in the connected component of 1. But then so is vw, because -1 is in this component.

Finally, if the restriction of q to Z is non-degenerate, set $\epsilon_1 = v$, and call ϵ_2 an element of Z of length 1 that is othogonal to v. We can write $w = a\epsilon_1 + b\epsilon_2$, with $a^2 + b^2 = 1$; then $vw = -a + b\epsilon_1\epsilon_2$, and vw is in the image of a map from the circle group, again by Remark 9.7.

This ends the proof of (d).

For (e), notice that since Spin_n has index 2 in Pin_n, the commutator subgroup of Pin_n is contained in Spin_n. For the reverse inclusion, we see from Theorem (c) that every element of Spin_n is the product of an even number of vectors of length 1; hence it is enough to show that every product of two vectors of length 1 is in the commutator subgroup.

Let us notice the following: if v and w are vectors of length 1, then

$$\begin{aligned}[v, w] &= vwv^{-1}w^{-1} \\ &= vw(-v)(-w) \\ &= (vw)^2.\end{aligned}$$

Consider a product vw, where v and w are vectors of length 1. If v and w are linearly dependent, then $w = \pm v$, and $vw = \mp 1$. We have that 1 is obviously a commutator, while $[e_1, e_2] = (e_1e_2)^2 = -1$.

Next, assume that v and w are linearly independent, and call Z the subspace of V that they generate. Assume that the restriction of the quadratic form $\langle - \mid - \rangle$ to Z is degenerate; let u be a generator of the radical of $q|_Z$. Then w is of the form $\pm v + \alpha u$. If $w = v + \alpha u$, then $vw = (-1)v(-w)$; since we know that -1 is a commutator, we may assume that $w = -v + \alpha u$. We have that $\left|-v + \frac{\alpha}{2}u\right|^2 = |v|^2 = 1$, and

$$\begin{aligned}\Big[v, -v + \frac{\alpha}{2}u\Big] &= \Big(v(-v + \frac{\alpha}{2}u)\Big)^2 \\ &= \Big(1 + \frac{\alpha}{2}vu\Big)^2 \\ &= 1 + \alpha vu \\ &= vw\end{aligned}$$

Now the case when v and w are linearly independent, and the restriction of $\langle - \mid - \rangle$ the subspace Z is non-degenerate. Set $\epsilon_1 = v$, and let ϵ_2 be a vector of Z that is orthogonal to ϵ_1, with $|\epsilon_2|^2 = 1$. Write $w = a\epsilon_1 + b\epsilon_2$; we have $a^2 + b^2 = 1$. Consider the subgroup S of elements of Spin_n of the form $\alpha + \beta\epsilon_1\epsilon_2$, with α, $\beta \in K$,

$\alpha^2 + \beta^2 = 1$, as in Remark 9.7. Then

$$\begin{aligned} vw &= \epsilon_1(a\epsilon_1 + b\epsilon_2) \\ &= -a + \epsilon_1\epsilon_2 \\ &\in S. \end{aligned}$$

Since S is isomorphic to either $\mathbb{S}^1$ or $\mathbb{C}^*$, it is a divisible group, so we can find $\alpha + \beta\epsilon_1\epsilon_2 \in S$ with $(\alpha\epsilon_1 + \beta\epsilon_2)^2 = -a + \epsilon_1\epsilon_2$. If we set $z = -\alpha\epsilon_1 + \beta\epsilon_2 \in Z$, we have that $vz = \alpha + \beta\epsilon_1\epsilon_2$; hence $[v, z] = (vz)^2 = vw$, and this ends the proof of (e).

More precisely, we have shown that Spin_n is generated by commutators of the form $[v, w]$, where v and w are vectors of length 1 in V. To prove (f) it is enough to show that every commutator of the form $[v, w]$ as above is a commutator of two elements of Spin_n. But since $n \geq 3$, given v and w in V we can choose a vector $z \in V$ that is orthogonal to v and w, and such that $|z|^2 = 1$. Then z anticommutes with v and w, hence

$$\begin{aligned} [vz, wz] &= vzwz(vz)^{-1}(wz)^{-1} \\ &= vzwzzvzw \\ &= -vzwvzw \\ &= -vzzwvw \\ &= vwvw \\ &= [v, w]; \end{aligned}$$

this ends the proof, since vz and wz are in Spin_n. ♠

We can also determine the Lie algebra of Spin_n. Since the homomorphism $\rho\colon \mathrm{Spin}_n \to \mathrm{SO}_n$ is surjective with finite kernel, ρ is a local diffeomorphism, and thus induces an isomorphism of the Lie algebra of Spin_n with the Lie algebra $\mathfrak{so}_n$ of SO_n. On the other hand, since Spin_n is a subgroup of the algebra $\mathrm{C}(V, q)$, its Lie algebra is a subalgebra of $\mathrm{C}(V, q) = \bigwedge^\bullet V$.

Proposition 9.13. *The Lie algebra of* Spin_n *is* $\bigwedge^2 V \subseteq \mathrm{C}(V, q)$.

This gives a more conceptual proof of Theorem 5.9.

Proof. Call L the Lie algebra of Spin_n. Since the dimension of Spin_n is $n(n-1)/2$, which is the dimension of $\bigwedge^2 V$, it is enough to prove that L is contained in $\bigwedge^2 V$.

Consider the equations that define Spin_n. Since Spin_n is contained in $\mathrm{C}^+(V, a)$, which is a linear subspace, L will be contained in $\mathrm{C}^+(V, a) = \bigwedge^+ V$.

The other conditions that define Spin_n are not linear, so we need to differentiate.

The conjugation map $a \mapsto \overline{a}$ from $\mathrm{C}(V, q)$ to itself is linear; hence by differentiating the function $a \mapsto a\overline{a}$ at the origin we get the map $x \mapsto x + \overline{x}$. So L is contained in the subspace of elements $x \in \bigwedge^\bullet V$ such that $x + \overline{x} = 0$. Since $\overline{x} = (-1)^{|x|(|x|+1)/2}$ and x is even, we see that the homogeneous components of x are of degree $2k$, where k is odd.

By differentiating the condition that $av\overline{a} \in V$ for all $v \in V$, we get that L is contained in the subspace L_0 of elements x of $\bigwedge^\bullet V$ of degree $2k$, with k odd, such that $xv - vx = xv + v\overline{x} \in V$ for all $v \in V$. If $x \in L_0$, since x has only components of even degree, we see that

$$\begin{aligned} xv - vx &= x \wedge v + x \dashv v - v \wedge x - v \vdash x \\ &= -2v \vdash x. \end{aligned}$$

If we write $x = x_2 + x_6 + x_{10} + \dots$ as a sum of components of various degrees, we have

$$v \vdash x_6 = v \vdash x_{10} = \cdots = 0$$

for all $v \in V$. By Lemma 3.4, since the form q is non-degenerate we have $x_6 = x_{10} = \cdots = 0$; so $L_0 \subseteq \bigwedge^2 V$. This show that $L \subseteq \bigwedge^2 V$, and concludes the proof. ♠

The center of the spin group. Let us determine the center of the spin group. This center contains at least the subgroup $\{\pm 1\}$. Consider the element $\eta \stackrel{\text{def}}{=} e_1 \dots e_n \in \mathrm{Pin}_n$; if n is even, then η is in Spin_n.

Theorem 9.14. *Assume that $n \geq 3$.*

If n is odd, then the center of Spin_n is $\{\pm 1\}$.

If n is even, the center of Spin_n is the subgroup $\{\pm 1, \pm \eta\}$. If $n \equiv 0 \pmod 4$, then η has order 2, and the center is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; if $n \equiv 2 \pmod 4$, then η has order 4, and the center is cyclic of order 4.

Proof. Since $\{\pm 1\}$ is contained in the center of Spin_n, the center of Spin_n is the inverse image in Spin_n of the center of SO_n. We will prove the well known and easy fact that if n is odd the center of SO_n is trivial, while when n is even consists of $\{\pm \mathrm{I}_n\}$. Then the first two statement follow, because the image of η in SO_n is the composition of the reflexions along all the coordinate hyperplanes, which is clearly $-\mathrm{I}_n$.

To prove this, let $A \in \mathrm{SO}_n$ be in the center. Notice that if $Z \subseteq \mathbb{C}^n$ is a 2-dimensional linear subspace such that $q|_Z$ is non-degenerate, then $\mathbb{C}^n = Z \oplus Z^\perp$, and we have an element B of SO_n that acts like the identity on $Z^\perp$ and like multiplication by -1 on Z. Since A commutes with B we see that A sends Z into itself.

Take $v \in \mathbb{C}^n$ such that $|v|^2 \neq 0$. Then since $n \geq 3$, we can find w_1 and w_2, such that $|w_1|^2 \neq 0$, $|w_2|^2 \neq 0$, and $\langle v \mid w_1 \rangle = \langle v \mid w_2 \rangle = \langle w_1 \mid w_2 \rangle = 0$. The restriction of q to the subspaces Z_1 and Z_2 generated respectively by $\{v, w_1\}$ and by $\{v, w_2\}$ is non-degenerate, hence A sends $\langle v \rangle = Z_1 \cap Z_2$ into itself. So A sends $e_1, \dots, e_n$ and $e_1 + \cdots + e_n$ into multiples of themselves, so it is a scalar matrix. But the only scalar matrices in SO_n are I_n, and $-\mathrm{I}_n$ if n is even.

To determine the structure of the center of Spin_n when n is even, notice that

$$\begin{aligned} \eta^2 &= (-1)^{\frac{n(n-1)}{2}} \eta \eta^{\mathrm{t}} \\ &= (-1)^{\frac{n(n-1)}{2}} e_1 \dots e_n e_n \dots e_1 \\ &= (-1)^{\frac{n(n-1)}{2}} (-1)^n \\ &= (-1)^{n/2} \end{aligned}$$

which is easily seen to imply the result. ♠

Remark 9.15. It is easy to see directly that η is in the center of Spin_n when n is even: the point is e_i anti-commutes with e_j when $i \neq j$, while it commutes with e_i

itself: hence $\eta e_i = (-1)^{n-1} e_i \eta = -e_i \eta$. So $\eta v = -v\eta$ for any $v \in V$, and the result follows, because every element of Spin_n is a product of a finite number of vectors of length 1.

THE SPIN REPRESENTATION

The group $\mathrm{Spin}_n(\mathbb{C})$ is contained the group of units in $\mathrm{C}^+(\mathbb{C}^n, q)$; hence every module over $\mathrm{C}^+(\mathbb{C}^n, q)$ gives a representation of $\mathrm{Spin}_n(\mathbb{C})$, and also of $\mathrm{Spin}_n(\mathbb{R})$, since $\mathrm{Spin}_n(\mathbb{R})$ is contained in $\mathrm{Spin}_n(\mathbb{C})$. The spinor space $\bigwedge^\bullet W$ has a structure of $\mathrm{C}^+(\mathbb{C}^n, q)$-module; when n is odd this is irreducible, while when n is even this splits as the sum of two half-spinor spaces $\bigwedge^+ W$ and $\bigwedge^- W$, each of which is irreeducible.

So we get the *spin representation* $\bigwedge^\bullet W$ of $\mathrm{Spin}_n(\mathbb{C})$ and $\mathrm{Spin}_n(\mathbb{R})$. When n is even the spin representation is the sum of two *half-spin representations* $\bigwedge^+ W$ and $\bigwedge^- W$. These are often called *chiral spin representations* by physicists.

Proposition 9.16. *If n is odd, the spin representation $\bigwedge^\bullet W$ of $\mathrm{Spin}(\mathbb{R})$ and of $\mathrm{Spin}_n(\mathbb{C})$ is irreducible. If n is even, then each of the two half-spin representations $\bigwedge^+ W$ and $\bigwedge^- W$ of $\mathrm{Spin}(\mathbb{R})$ and of $\mathrm{Spin}_n(\mathbb{C})$ is irreducible.*

Proof. Suppose that n is odd; we know that the spin $\mathrm{C}^+(\mathbb{C}^n, q)$-module $\bigwedge^\bullet W$ is simple. If Z is a subspace of $\bigwedge^\bullet W$, the set elements of $\mathrm{C}^+(\mathbb{C}^n, q)$ that send Z into itself form a subalgebra of $\mathrm{C}^+(\mathbb{C}^n, q)$. Since the elements of $\mathrm{C}(\mathbb{C}^n, q)$ of the form $e_i e_j$ are in $\mathrm{Spin}_n(\mathbb{R}) \subseteq \mathrm{Spin}_n(\mathbb{C})$ and generate $\mathrm{C}^+(\mathbb{C}^n, q)$, the first statement follows.

The second statement is proved analogously, because we know that each of the two half-spin $\mathrm{C}^+(\mathbb{C}^n, q)$-modules is simple. ♠

Examples 9.17. Let us work out the structure of the spin and half-spin representations when $n = 2$ and $n = 3$. We rely on our analysis of the structures of Spin_2 and Spin_3 carried out in Examples 9.6 and 9.8.

In these cases W is 1-dimensional, generated by $w_1 = \frac{1}{2}(\mathrm{i}e_1 + e_2)$, while W' is generated by $w'_1 = \frac{1}{2}(\mathrm{i}e_1 - e_2)$. The spinor space $\bigwedge^\bullet W = \mathbb{C} \oplus W$ has a basis $1, w_1$; we will write all the endomorphisms of $\bigwedge^\bullet W$ as matrices relative to these basis. The half-spinor spaces are $\bigwedge^+ W = \mathbb{C}$ and $\bigwedge^- W = W$.

(a) Assume that $n = 2$. Then w_1 acts on $\bigwedge^\bullet W$ by sending 1 to w_1 and w_1 to 0, while w'_1 sends 1 to 0; thus under the isomorphism $\mathrm{C}(\mathbb{C}^2, 2) \simeq \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W) = \mathrm{M}_2(\mathbb{C})$ the element w_1 and w'_1 correspond to the matrices

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $e_1 = -\mathrm{i}(w_1 + w'_1)$ and $e_2 = w_1 - w'_1$, we see that e_1 and e_2 correspond to

$$\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence $e_1 e_2$ acts like the matrix

$$\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix};$$

an element of Spin_2, which is of the form $a + be_1e_2$, with $a, b \in K$ with $a^2 + b^2 = 1$, acts on the spinor space by multplication by $a + \mathrm{i}b$ on $\bigwedge^+ W = \mathbb{C}$, and by multiplication by $a - \mathrm{i}b = (a + \mathrm{i}b)^{-1}$ on $\bigwedge^- W = W$. Since the isomorphisms $\mathrm{Spin}_2(\mathbb{R}) \simeq \mathbb{S}^1$ and $\mathrm{Spin}_2(\mathbb{C}) \simeq \mathbb{C}^*$ are obtained precisely by sending $a + be_1e_2$ into $a + \mathrm{i}b$, we see that the even half-spin representation is the natural 1-dimensional of the real or complex circle group by multiplication, while the odd half-spin representation is its dual, multiplication by the inverse.

(b) Now we assume $n = 3$. The elements e_1 and e_2 act as in the previous case, while $u_0 = \mathrm{i}e_3$ acts via the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From this we see that element $\epsilon_1 = e_2e_3$, $\epsilon_2 = e_3e_1$ and $\epsilon_3 = e_1e_2$ correspond to the matrices

$$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix};$$

hence an element of Spin_3, which is written as $a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3$, where the a_i are scalars satisfying $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$, corresponds to the matrix

$$\begin{pmatrix} a_0 + \mathrm{i}a_3 & -a_1 + \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & a_0 - \mathrm{i}a_3 \end{pmatrix}.$$

The determinant of this matrix is precisely $a_0^2 + a_1^2 + a_2^2 + a_3^2$; so we see how the spin representation yields an isomorphism of $\mathrm{Spin}_3(\mathbb{C})$ with $\mathrm{SL}_2(\mathbb{C})$. Under this isomorphism, the elements of $\mathrm{Spin}_3(\mathbb{R})$ correspond to matrices of the form above where all the a_i are real; but it is easy to see how these matrices are precisely the unitary matrices of determinant 1. So the spin representation also gives an isomorphism of $\mathrm{Spin}_3(\mathbb{R})$ with SU_2.

Proposition 9.18. *Assume that $n \geq 3$. Then the spin representation $\mathrm{Spin} \to \mathrm{GL}(\bigwedge^\bullet W)$ factors through $\mathrm{SL}(\bigwedge^\bullet W)$. Furthermore, if n is even each of the half-spin representations $\mathrm{Spin}_n \to \mathrm{GL}(\bigwedge^+ W)$ and $\mathrm{Spin}_n \to \mathrm{GL}(\bigwedge^- W)$ factor through $\mathrm{SL}(\bigwedge^+ W)$ and $\mathrm{SL}(\bigwedge^- W)$ respectively.*

Proof. This follows from Theorem 9.12 (f). ♠

Corollary 9.19. *We have isomorphisms of algebraic groups*

$$\mathrm{Spin}_3(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C})$$

and

$$\mathrm{Spin}_4(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}).$$

Proof. For the first one, $\mathrm{Spin}_3(\mathbb{C})$ is embedded in the group of units of $\mathrm{C}(\mathbb{C}^3, q) = \mathrm{End}_{\mathbb{C}}(\bigwedge^\bullet W)$, which is $\mathrm{GL}_2(\mathbb{C})$; because of Proposition 9.18 we have that Spin_3 is embedded in $\mathrm{SL}_2(\mathbb{C})$. Since $\mathrm{Spin}_3(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ both have dimension 3, and $\mathrm{SL}_2(\mathbb{C})$ is connected, the result follows.

For the second one the argument is similar: $\mathrm{Spin}_4(\mathbb{C})$ is embedded in the group of units of $\mathrm{C}^+(\mathbb{C}^4, q) = \mathrm{End}_{\mathbb{C}}(\bigwedge^+ W) \times \mathrm{End}_{\mathbb{C}}(\bigwedge^- W)$, which is $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, ; because of Proposition 9.18 it is embedded in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. Since both groups have dimension 6 and $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is connected, the conclusion follows. ♠

Notice the isomorphism $\mathrm{Spin}_3(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C})$ has been constructed direcly in Example 9.8.

There are other "exceptional" isomorphisms of Spin_5 and Spin_6 with other known algebraic groups; these require the introduction of some structure on the spinor space.

The canonical pairings: the even case. Now we assume that n is even, $n = 2m$.

We have defined the canonical pairings β, $\overline{\beta}\colon \bigwedge^\bullet W \times \bigwedge^\bullet W \to \mathbb{C}$, which are symmetric or alternating depending on m (Proposition 8.16). Recall if a group G acts linearly on a K-vector space Z, a bilinear pairing $h\colon Z \times Z \to K$ is said to be *invariant under G* if $h(gx, gy) = h(x, y)$ for all $g \in G$ and all $x, y \in Z$.

Theorem 9.20. *The form β is invariant under the action of Spin_n, while $\overline{\beta}$ is invariant under the action of Pin_n.*

Proof. Here is the basic fact, that will be also be very useful later.

Lemma 9.21. *For any $v \in V$, x and $y \in \bigwedge^\bullet W$, we have the relations*

$$\beta(vx, y) = \beta(x, vy), \qquad \overline{\beta}(vx, y) = -\overline{\beta}(x, vy)$$

and

$$\beta(vx, vy) = -|v|^2 \beta(x, y), \quad \overline{\beta}(vx, vy) = |v|^2 \overline{\beta}(x, y).$$

Proof. Let us prove that $\beta(vx, y) = \beta(x, vy)$. We may assume that v is either in W or in W'. In the first case, $vx = v \wedge x$, and the result is immediate:

$$\begin{aligned}\beta(vx, y) &= \int (v \wedge x)^{\mathrm{t}} \wedge y \\ &= \int x^{\mathrm{t}} \wedge v \wedge y \\ &= \beta(x, vy).\end{aligned}$$

If $v \in W'$ then $vx = 2v \vdash x$, and the proof is a little more elaborate. We may assume that x and y are homogeneous. Notice that both sides of the equality are zero unless $|x| + |y| = m + 1$; and then we have $x^{\mathrm{t}} \wedge y = 0$, so

$$\begin{aligned}0 &= v \vdash (x^{\mathrm{t}} \wedge y) \\ &= (v \vdash x^{\mathrm{t}}) \wedge y + (-1)^{|x|} x^{\mathrm{t}} \wedge (v \vdash y) \\ &= (-1)^{|x|-1}\big((x^{\mathrm{t}} \vdash v) \wedge y - x^{\mathrm{t}} \wedge (v \vdash y)\big) \\ &= (-1)^{|x|-1}\big((v \vdash x)^{\mathrm{t}} \wedge y - x^{\mathrm{t}} \wedge (v \vdash y)\big).\end{aligned}$$

By integrating we obtain the equality

$$\begin{aligned}\beta(vx, y) &= 2\int (v \vdash x)^{\mathrm{t}} \wedge y \\ &= 2\int x^{\mathrm{t}} \wedge (v \vdash y) \\ &= \beta(x, vy).\end{aligned}$$

The other equalities follow from the first. We have

$$\begin{aligned}\beta(vx, vy) &= \beta(x, v^2xy)\\ &= q(v)\beta(x,y)\\ &= -|v|^2\beta(x,y)\end{aligned}$$

and

$$\begin{aligned}\overline{\beta}(vx, y) &= \beta\big(\epsilon(vx), y\big)\\ &= -\beta\big(v\epsilon(x), y\big)\\ &= -\beta\big(\epsilon(x), vy\big)\\ &= -\overline{\beta}(x, vy).\end{aligned}$$

The argument for the fourth equality is similar. ♠

The theorem follows immediately from Lemma 9.21, and from the fact that every element of Pin_n is a product of vectors of length 1, while every element of Spin_n is a product of an even number of vectors of length 1. ♠

The situation is rather different according to whether m is even or odd. Let us assume that m is even, that is, $n \equiv 0 \pmod 4$. Then by construction the pairings β and $\overline{\beta}$ vanish on $\bigwedge^+ W \times \bigwedge^- W$ and on $\bigwedge^- W \times \bigwedge^+ W$, and give non-degenerate pairings

$$\bigwedge^+ W \times \bigwedge^+ W \longrightarrow \mathbb{C} \quad \text{and} \quad \bigwedge^- W \times \bigwedge^- W \longrightarrow \mathbb{C};$$

these pairings are either symmetric or alternating according to whether $m \equiv 0$ or $m \equiv 2 \pmod 4$ (Proposition 8.16).

Since by construction Spin_n is contained in the group of units of $\mathrm{C}^+(\mathbb{C}^n, q) = \mathrm{End}(\bigwedge^+ W) \times \mathrm{End}(\bigwedge^- W)$, which is $\mathrm{GL}(\bigwedge^+ W) \times \mathrm{GL}(\bigwedge^- W)$, from this and from Proposition 9.18 we get the following.

Proposition 9.22. *When $n \equiv 0 \pmod 8$ the two half-spin representations of* Spin_n *yield an embedding*

$$\mathrm{Spin}_n \subseteq \mathrm{SO}_{2^{m-1}}(\mathbb{C}) \times \mathrm{SO}_{2^{m-1}}(\mathbb{C});$$

while if $n \equiv 4 \pmod 8$ they give an embedding

$$\mathrm{Spin}_n \subseteq \mathrm{Sp}_{2^{m-2}}(\mathbb{C}) \times \mathrm{Sp}_{2^{m-2}}(\mathbb{C}).$$

In the case $n = 4$ we have $\mathrm{Sp}_1(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$, and we recover the isomorphism $\mathrm{Spin}_4(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ of Corollary 9.19.

In the next case, $n = 8$, Spin_8 is embedded into $\mathrm{SO}_8 \times \mathrm{SO}_8$; the dimension of Spin_8 is 28 while that of $\mathrm{SO}_8 \times \mathrm{SO}_8$ is 56, so we don't get an isomorphism. However, something remarkable happens: the standard representation and the two half-spin representations of Spin_8 are all orthogonal representations of dimension 8. These three representations are related via *triality*, which will be treated later.

When m is odd, we see from Proposition 8.16 that one between β and $\overline{\beta}$ is symmetric, while the other is alternating. The fact that the spin representation preserves both a symmetric and an alternating form may seem striking, but it is in fact rather mundane. We see from the definitions that wen m is odd β and $\overline{\beta}$ both vanish on $\bigwedge^+ W \times \bigwedge^+ W$ and $\bigwedge^- W \times \bigwedge^- W$. Both β and $\overline{\beta}$ give non-degenerate pairings

$$\bigwedge^+ W \times \bigwedge^- W \longrightarrow \mathbb{C} \quad \text{and} \quad \bigwedge^- W \times \bigwedge^+ W \longrightarrow \mathbb{C};$$

β and $\overline{\beta}$ agree on $\bigwedge^+ W \times \bigwedge^- W$, while the differ by the sign on $\bigwedge^- W \times \bigwedge^+ W$. Furthermore, the second paring is simply the transpose of the first, up to sign. Hence the only essential piece of information that comes from β and $\overline{\beta}$ is a non-degenerate paring $\bigwedge^+ W \times \bigwedge^- W \to \mathbb{C}$, that is invariant under the action of Spin_n.

Proposition 9.23. *Assume that $n \equiv 2 \pmod 4$, and write $n = 2m$. Then the two half spin representations are dual to each other and faithful.*

Furthermore, if $n \geq 6$ each half spin representation gives an embedding

$$\mathrm{Spin}_n \subseteq \mathrm{SL}_{2^{m-1}}(\mathbb{C}).$$

Proof. The fact that the two half-spin representations are dual follows from the existence of the invariant pairing $\bigwedge^+ W \times \bigwedge^- W \to \mathbb{C}$ constructed above.

Since Spin_n is contained in $\mathrm{GL}(\bigwedge^+ W) \times \mathrm{GL}(\bigwedge^- W)$, and since the kernel of the two half-spin representations $\mathrm{Spin}_n \to \mathrm{GL}(\bigwedge^+ W)$ and $\mathrm{Spin}_n \to \mathrm{GL}(\bigwedge^- W)$ are the same, since the two representations are dual, we see that the half-spin representations are both faithful.

The last statement follows from this and from Proposition 9.18. ♠

Corollary 9.24. *As an algebraic group, $\mathrm{Spin}_6(\mathbb{C})$ is isomorphic to $\mathrm{SL}_4(\mathbb{C})$.*

Proof. We have seen that there is an embedding of algebraic groups $\mathrm{Spin}_6(\mathbb{C}) \subseteq \mathrm{SL}_4(\mathbb{C})$. Both groups are 15-dimensional and $\mathrm{SL}_4(\mathbb{C})$ is connected, so the result follows. ♠

The canonical pairings: the odd case. Now we assume that n is odd, $n = 2m+1$. The situation is different from the even case: in each dimension, only one of the two form β and $\overline{\beta}$ is invariant under the action of Spin_n.

Theorem 9.25. *If $n \equiv 1 \pmod 4$, then the bilinear form β on $\bigwedge^\bullet W$ is invariant under the action of Spin_n.*

If $n \equiv 3 \pmod 4$, then the bilinear form $\overline{\beta}$ on $\bigwedge^\bullet W$ is invariant under the action of Spin_n.

Proof. We have an analogue of Lemma 9.21.

Lemma 9.26. *3 Suppose that $v \in V$, $x, y \in \bigwedge_1^\bullet W$. If m is even, then*

$$\beta(vx, y) = \beta(x, vy) \qquad \text{and} \quad \beta(vx, vy) = -|v|^2 \beta(x, y);$$

while

$$\overline{\beta}(vx, y) = -\overline{\beta}(x, vy) \quad \text{and} \quad \overline{\beta}(vx, vy) = |v|^2 \overline{\beta}(x, y)$$

if m is odd.

♠

Proof. The formulas for $\beta(vx, vy)$ and $\overline{\beta}(vx, vy)$ follow from the preceding ones, as in the proof of Lemma 9.21.

To prove the formulas for $\beta(vx, y)$ and $\overline{\beta}(vx, y)$, from Lemma 9.21 we know that they are correct when v is in $W \oplus W'$; since every vector in V is the sum of a vector in $W \oplus W'$ and a multiple of u_0, we may assume that $v = u_0$.

We may assume that x and y are homogeneous; and then $\beta(u_0x, y)$ and $\overline{\beta}(u_0x, y)$ are 0, unless $|x| + |y| = m$, in which case $(-1)^{|x|} = (-1)^m(-1)^{|y|}$. From this we get $\beta(u_0x, y) = (-1)^m\beta(x, u_0y)$ and $\overline{\beta}(u_0x, y) = (-1)^m\overline{\beta}(x, u_0y)$, and we are done. ♠

Putting this together with Proposition 8.16 we get the following.

Proposition 9.27. *If $n \equiv 1$ or $n \equiv 7 \pmod 4$ the spin representation gives an embedding*

$$\mathrm{Spin}_n \subseteq \mathrm{SO}_{2^m};$$

while if $n \equiv 3$ or $n \equiv 5 \pmod 5$ it gives an embedding

$$\mathrm{Spin}_n \subseteq \mathrm{Sp}_{2^{m-1}}.$$

Corollary 9.28. *As an algebraic group, $\mathrm{Spin}_5(\mathbb{C})$ is isomorphic to $\mathrm{Sp}_2(\mathbb{C})$.*

The exceptional isomorphisms. Let us collect the result about the structure of $\mathrm{Spin}_n(\mathbb{C})$ for $n \leq 6$ that have proved in Examples 9.5 and 9.6, and in Corollaries 9.19, 9.24 and 9.28.

Theorem 9.29. *We have the following isomorphisms of algebraic groups:*

$$\begin{aligned}
&\mathrm{Spin}_1(\mathbb{C}) \simeq \mathbb{Z}/2\mathbb{Z},\\
&\mathrm{Spin}_2(\mathbb{C}) \simeq \mathbb{C}^*\\
&\mathrm{Spin}_3(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}),\\
&\mathrm{Spin}_4(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}),\\
&\mathrm{Spin}_5(\mathbb{C}) \simeq \mathrm{Sp}_2(\mathbb{C}),\\
&\mathrm{Spin}_6(\mathbb{C}) \simeq \mathrm{SL}_4(\mathbb{C}).
\end{aligned}$$

Remark 9.30. The structure theory of algebraic groups shows that no isomorphism of this kind can exist for $n \geq 7$, because then the Dynkin diagram of Spin_n does not coincide with the Dynkin diagram of any other simply connected algebraic group.

10. Triality

From now on the base field will be $\mathbb{C}$; we will write Spin_n and SO_n for $\mathrm{Spin}_n(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{C})$. We will also denote the two half-spin modules $\bigwedge^+ W$ and $\bigwedge^- W$ by S^+ and S^- respectively.

The group Spin_8 has three 8-dimensional orthogonal representations: the representation $\rho\colon \mathrm{Spin}_8 \to \mathrm{SO}_8$, and the two half-spin representations, denoted by $\sigma^+\colon \mathrm{Spin}_8 \to \mathrm{GL}(S^+)$ and $\sigma^-\colon \mathrm{Spin}_8 \to \mathrm{GL}(S^-)$ (we think of the two half-spin spaces as endowed with the non-degenerate quadratic form $\overline{\beta}$, which is invariant under Spin_8, according to Theorem 9.20).

Hence we get three representations ρ, σ^+ and $\sigma^-\colon \mathrm{Spin}_8 \to \mathrm{SO}_8$. These are not isomorphic: this can be checked, for example, by looking at the action of the center $\{\pm 1, \pm\eta\}$ of Spin_8, where $\eta = e_1 \dots e_8$ (Theorem 9.14). In fact $-1 \in \mathrm{Spin}_8$ is sent to the identity by ρ, while it acts as multiplication by -1 on S^+ and S^-. So neither σ^+ nor σ^- is isomorphic to ρ.

To see that σ^+ and σ^- are not isomorphic, we need to see how η acts on S^+ and S^-. Notice that, since σ^+ and σ^- are irreducible representations of Spin_8, by Proposition 9.16, η must act as a scalar, by Schur's Lemma. We have

$$e_i = -\sqrt{-1}(w_i + w_i')$$

for $1 \leq i \leq 4$ and

$$e_i = w_{i-4} - w_{i-4}'$$

for $5 \leq i \leq 8$; hence

$$\eta = (w_1 + w_1')(w_2 + w_2')(w_3 + w_3')(w_4 + w_4')(w_1 - w_1')(w_2 - w_2')(w_3 - w_3')(w_4 - w_4').$$

An elementary calculation, using the definition of the action of Spin_8 on the space of spinor, tells us that $\eta 1 = 1$, while $\eta w_4 = -w_4$; hence η acts like the identity on S^+ and as multiplication by -1 on S^-. So σ^+ and σ^- are not isomorphic.

What is the relation among these three representations? One way to state it is the following.

Theorem 10.1. *There exists an action of the symmetric group* S_3 *on the algebraic group* Spin_8 *acting as the full symmetric group on the set consisting of the isomorphism classes of these three representations.*

The simple-minded construction that comes to mind is the following. If we choose isomorphisms of quadratic forms $(\mathbb{C}^8, q) \simeq (S^+, \beta) \simeq (S^-, \beta)$ we obtain an embedding $\mathrm{Spin}_8 \subseteq \mathrm{SO}_8 \times \mathrm{SO}_8 \times \mathrm{SO}_8$; there is an obvious action of S^3 on $\mathrm{SO}_8 \times \mathrm{SO}_8 \times \mathrm{SO}_8$ by permuting the components, and one might expect that by chosing the isomorphisms in the right way one can have Spin_8 to be invariant in the product.

This can not work. In fact, since the first projection $\mathrm{Spin}_8 \to \mathrm{SO}_8$ is a topological cover, there is no non-trivial automorphism of Spin_8 over the identity in SO_8. The action of S_3 on Spin_8 does come from an action of S_3 on $\mathrm{SO}_8 \times \mathrm{SO}_8 \times \mathrm{SO}_8$, but it is not the obvious one. It is constructed by means of an amazing piece of linear algebra, known as *triality*. Triality is like duality, only it involves three vector spaces instead on two; duality between two vector spaces is defined by a bilinear form, triality is defined by a trilinear form.

Let (V_1, q_1) and (V_2, q_2) be two non-degenerate quadratic forms; these give isomorphisms $V_i \simeq V_i^\vee$, defined by sending $v \in V_i$ into the linear form $q_i(v, -)$. Recall that there is a canonical isomorphism $\mathrm{Hom}(V_1, V_2) \simeq \mathrm{Hom}(V_2, V_1)$, in which any linear map $f\colon V_1 \to V_2$ corresponds to its transpose $f^{\mathrm{t}}\colon V_2 \to V_1$, defined by the equality

$$q_2\bigl(f(v_1), v_2\bigr) = q_1\bigl(v_1, f^{\mathrm{t}}(v_2)\bigr)$$

for any $v_1 \in V_1$ and $v_2 \in V_2$. Furthermore $\mathrm{Hom}(V_1, V_2)$ and $\mathrm{Hom}(V_2, V_1)$ are also isomorphism to the space of bilinear forms $V_1 \times V_2 \to \mathbb{C}$; a bilinear form $\Phi\colon V_1 \times V_2 \to \mathbb{C}$ corresponds to the linear maps $f_1\colon V_1 \to V_2$ and $f_2\colon V_2 \to V_1$ defined by

$$\begin{aligned}\Phi(v_1, v_2) &= q_2\bigl(f_1(v_1), v_2\bigr)\\ &= q_1\bigl(v_1, f_2(v_2)\bigr).\end{aligned}$$

for any $v_1 \in V_1$ and $v_2 \in V_2$. Clearly f_1 and f_2 are the transpose of each other.

The following is very easy.

Proposition 10.2. *Let* $\Phi\colon V_1 \times V_2 \to \mathbb{C}$ *be a bilinear map, corresponding to the linear maps* $f_1\colon V_1 \to V_2$ *and* $f_2\colon V_2 \to V_1$*. Then the following are equivalent.*

(a) f_1 *and* f_2 *are isometric maps.*
(b) *Either* f_1 *or* f_2 *is an isometric map, and* $\dim V_1 = \dim V_2$.

Such a bilinear map Φ defines a *duality* between the two quadratic forms.

Now consider three non-degenerate quadratic forms (V_1, q_1), (V_2, q_2) and (V_3, q_3). A bilinear map $f\colon V_1 \times V_2 \to V_3$ is called *orthogonal* when the equality

$$q_3\big(f(v_1, v_2)\big) = q_1(v_1)q_2(v_2)$$

holds for any $v_1 \in V_1$ and $v_2 \in V_2$.

A trilinear form $\Phi\colon V_1 \times V_2 \times V_3 \to \mathbb{C}$ gives a bilinear map $V_1 \times V_2 \to V_3$, denoted by $(v_1, v_2) \mapsto v_1v_2$, defined by the formula

$$q_3\big(v_1v_2, v_3\big) = \Phi(v_1, v_2, v_3).$$

This gives an isomorphism of the space of trilinear forms $V_1 \times V_2 \times V_3 \to \mathbb{C}$ with the space of bilinear maps $V_1 \times V_2 \to V_3$. But of course the same holds for any two distinct indices i and j between 1 and 3: if $\{1, 2, 3\} = \{i, j, k\}$, the trilinear form Φ yields bilinear maps $V_i \times V_j \to V_k$, denoted by $(v_i, v_j) \mapsto v_iv_j$.

This notation by juxtaposition should not give rise to confusion: in no case we will have more than one trilinear map around simultaneosly. It is of course ambiguous: for example, when $V_1 = V_2 = V_3 = V$, there are three possibly different bilinear maps $V \times V \to V$ denoted with the same notation. In this case one should probably make the V_i disjoint, for example by setting $V_i = V \times \{i\}$. In our main example the V_i will be distinct.

It is also easy to see how the various maps $V_i \times V_j \to V_k$ determine one other. For example, if $v_i \in V_i$ for $i = 1, 2$ and 3, we have

$$\begin{aligned} q_3(v_1v_2, v_3) &= \Phi(v_1, v_2, v_3) \\ &= q_2(v_2, v_1v_3); \end{aligned}$$

this can be read as saying that the linear functions $V_2 \to V_3$, $v_2 \mapsto v_1v_2$ and $V_3 \to V_2$, $v_3 \mapsto v_1v_3$, are the transpose of each other.

Furthermore, the bilinear functions $V_i \times V_j \to V_k$ have an obvious symmetry property: if $v_i \in V_i$ and $v_j \in V_j$, we have

$$v_iv_j = v_jv_i \in V_k.$$

These facts will be used without comments in what follows.

Proposition 10.3. *Let* $\Phi\colon V_1 \times V_2 \times V_3 \to \mathbb{C}$ *be a trilinear form, and assume that none of the* V_i *is* 0*. Then the following two conditions are equivalent.*

(a) *Each of the* $V_i \times V_j \to V_k$ *is orthogonal.*
(b) *One of the* $V_i \times V_j \to V_k$ *is orthogonal, and* $\dim V_1 = \dim V_2 = \dim V_3$.

Proof. Assume that $V_1 \times V_2 \to V_3$ is orthogonal. Since $V_1 \neq 0$, we can choose $v_1 \in V_1$ such that $q_1(v_1) = 1$. Then it follows from the definition of an orthogonal bilinear map that the map $V_2 \to V_3$ defined by $v_2 \mapsto v_1v_2$ is isometric, hence injective. So $\dim V_2 \leq \dim V_3$, and, symmetrically, $\dim V_3 \leq \dim V_2$; thus $\dim V_2 = \dim V_3$. If $V_1 \times V_3 \to V_2$ is also orthogonal we see that $\dim V_1 = \dim V_2 = \dim V_3$.

Conversely, assume that, for example, $V_1 \times V_2 \to V_3$ is orthogonal, and the dimensions are equal; let us show that, for example, $V_1 \times V_3 \to V_2$ is orthogonal. We need to prove the formula $q_2(v_1v_3) = q_1(v_1)q_3(v_3)$ for any $v_1 \in V_1$ and $v_3 \in V_3$. By continuity, it is enough to prove it assuming that $q_1(v_1) \neq 0$.

The two maps $f\colon V_2 \to V_3$ and $g\colon V_3 \to V_2$ defined by $f(v_2) \stackrel{\text{def}}{=} v_1v_2$ and $g(v_3) = v_1v_3$ are the transpose of each other. We have that $q_3\big(f(v_2)\big) = q_1(v_1)q_2(v_2)$ for any $v_2 \in V_2$; this implies that

$$\begin{aligned} q_3\big(gf(v_2), v_2'\big) &= q_3\big(f(v_2), f(v_2')\big) \\ &= q_1(v_1)q_2(v_2, v_2') \end{aligned}$$

for any v_2 and v_2' in V_2; hence $gf = q_1(v_1)\mathrm{id}_{V_2}$. Since V_2 and V_3 have the same dimension and $q_1(v_1) \neq 0$, this implies that $fg = q_1(v_1)\mathrm{id}_{V_2}$, so that

$$\begin{aligned} q_2(v_1v_3) &= q_2\big(g(v_3), g(v_3)\big)\big) \\ &= q_3\big(fg(v_3), v_3\big) \\ &= q_1(v_1)q_3(v_3). \end{aligned}$$

♠

Remark 10.4. It follows from the proof that if any two of the $V_i \times V_j \to V_k$ are orthogonal, then the dimensions of all the V_i are equal, so they are all orthogonal.

Definition 10.5. A *triality* is a sequence of three non-degenerate quadratic forms (V_1, q_1), (V_2, q_2) and (V_3, q_3), with V_1, V_2 and V_3 positive-dimensional, and a trilinear form $\Phi\colon V_1 \times V_2 \times V_3 \to \mathbb{C}$ satisfying the equivalent conditions of Proposition 10.3.

The *dimension* of such a triality is the common dimension of the V_i.

We will usually denote such a triality by (V_1, V_2, V_3, Φ), omitting the q_i from the notation.

By Proposition 10.3, to produce a triality is it enough to have three non-degenerate quadratic forms (V_1, q_1), (V_2, q_2) and (V_3, q_3) of the same positive dimension, and an orthogonal bilinear map $V_1 \times V_2 \to V_3$.

While dualities are extremely common, trialities are exceedingly rare. Let us start with some elementary example, which do not require the theory of Clifford algebras.

Examples 10.6. In all these examples we will have $V_1 = V_2 = V_3 = V$ and $q_1 = q_2 = q_3 = q$.

(a) Here V is $\mathbb{C}$, the quadratic form is the standard one $q(x) = x^2$, and the orthogonal map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the product $(x, y) \mapsto xy$.
(b) Now $V = \mathbb{C}^2$, the quadratic form is the hyperbolic form $q(x) = x_1x_2$, and orthogonal map $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ is given by $\big((x_1, x_2), (y_1, y_2)\big) \mapsto (x_1y_1, x_2, y_2)$.
(c) V is the space $\mathrm{M}_2(\mathbb{C})$ of 2×2 matrices, the quadratic form is the determinant, and the orthogonal map $\mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C}) \to \mathrm{M}_2(\mathbb{C})$ is the matrix product.

The next one is our basic example of triality; it is much more subtle than the preceding ones.

Example 10.7. We take $V_1 = \mathbb{C}^8$, $V_2 = S^+$, $V_3 = S^-$. The quadratic forms are $q(x) = -|x|^2$ for $\mathbb{C}^8$, and the restrictions of β for S^+ and S^-. The bilinear function

$\mathbb{C}^8 \times S^+ \to S^-$ is given by the $\mathrm{C}(\mathbb{C}^8, q)$-module structure on $\bigwedge^\bullet W = S^+ \oplus S^-$. Explicitly, the formula is that in the statement of Theorem 8.2.

We will denote by $\Phi\colon \mathbb{C}^8 \times S^+ \times S^- \to \mathbb{C}$ the resulting trilinear function.

The fact that this is triality follows from the equality $\beta(vx, vx) = q(x)\beta(x, x)$ for $v \in \mathbb{C}^8$ and $x \in S^+$ (Lemma 9.21). Also from Lemma 9.21 we know that $\beta(vx, y) = \beta(x, vy)$ when $v \in \mathbb{C}^8$, $x \in X^+$ and $y \in S^-$, which says that, given $v \in \mathbb{C}^8$, the traspose of $S^+ \to S^-$, $x \mapsto vx$, is $S^- \to S^+$, $y \mapsto vy$. In other words, the bilinear function $\mathbb{C}^8 \times S^- \to S^+$ determined by Φ is once again given by the spin module structure on $\bigwedge^\bullet W$.

It is a remarkable fact that we have constructed essentially all the examples of triality that exist. Here is a weaker version of this fact.

Theorem 10.8 (Hurwitz). *A triality can only have dimension* 1, 2, 4 *or* 8.

Proof. We start with a Lemma.

Lemma 10.9. *Let (V_1, V_2, V_3, Φ) be a triality. If $(i, j, k) = (1, 2, 3)$, v_i and v_i' are in V_i, v_j is in V_j, we have*

$$v_i(v_i' v_j) + v_i'(v_i v_j) = 2q_i(v_i, v_i')v_j.$$

and

$$v_i(v_i v_j) = q_i(v_i)v_j.$$

Proof. The two formulas are equivalent. The second one follows from the first by taking $v_i = v_i'$; the first one follows from the second by polarization. So it is enough to prove the second one: and the argument is given in the course of the proof of Proposition 10.3. ♠

There is a linear function $V_1 \to \mathrm{End}_{\mathbb{C}}(V_2 \oplus V_3)$ obtained by sending v_1 into $(v_2, v_3) \mapsto (v_1 v_3, v_1 v_2)$. Consider the Clifford algebra $\mathrm{C}(V_1, q_1)$, which is isomorphic to $\mathrm{C}(\mathbb{C}^n, q)$: the equality $v_1(v_1 v_j) = q_1(v_1)v_j$ (from Lemma 10.9) implies that this linear function extends to a homomorphism of $\mathbb{C}$-algebras $\mathrm{C}(V_1, q_1) \to \mathrm{End}_{\mathbb{C}}(V_2 \oplus V_3)$; thus $V_2 \oplus V_3$ is a $\mathrm{C}(V_1, q_1)$-module.

Since the elements of V_1 switch the two factors V_2 and V_3, these factors will be invariant under the even part $\mathrm{C}^+(V_1, q_1)$; hence V_2 and V_3 are modules over $\mathrm{C}^+(V_1, q_1)$; because of the structure of $\mathrm{C}^+(V_1, q_1)$ (Theorems 8.7 and Theorem 8.14), and of the structure of modules over products of matrix algebras (Proposition A.9) we see that V_2 is a direct sum of copies of $\bigwedge^\bullet W$ (in the odd case) or of $\bigwedge^+ W$ and $\bigwedge^- W$ (in the even case).

If we call n the common dimension of the V_i, and we set $n = 2m$ (if n is even) or $n = 2n + 1$ (in n is odd), we see that n is a multiple of 2^{m-1} (when n is even) or of 2^n (when n is odd). It is an elementary exercise to prove that this implies $n = 1, 2$, 4 or 8. ♠

Definition 10.10. Let $\tau = (V_1, V_2, V_3, \Phi)$ be a triality. A *restricted automorphism* of τ is triple (f_1, f_2, f_3), in which each f_i is an orthogonal automorphism of V_i, such that

$$\Phi\bigl(f_1(v_1), f_2(v_2), f_3(v_3)\bigr) = \Phi(v_1, v_2, v_3)$$

for any $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$.

The set of such restricted automorphisms, with the group structure given by component-wise composition, is called the *restricted automorphism group* of τ, and denoted by $\mathrm{Aut}^0(\tau)$.

Thus, $\mathrm{Aut}^0(\tau)$ is a subgroup of $\mathrm{O}(V_1, q_1) \times \mathrm{O}(V_2, q_2) \times \mathrm{O}(V_3, q_3)$.

Proposition 10.11. *Let $\tau = (V_1, V_2, V_3, \Phi)$ be a triality, (f_1, f_2, f_3) a triple in which each f_i is an orthogonal automorphism of V_i. This is a restricted automorphism if and only if $f_1(v_1)f_2(v_2) = f_3(v_1v_2)$ for any $v_1 \in V_1$ and $v_2 \in V_2$.*

Proof. We have

$$\Phi\big(f_1(v_1), f_2(v_2), f_3(v_3)\big) = q_3\big(f_1(v_1)f_2(v_2), f_3(v_3)\big)$$

and

$$\begin{aligned}\Phi(v_1, v_2, v_3) &= q_3(v_1v_2, v_3)\\ &= q_3\big(f_3(v_1v_2), f_3(v_3)\big)\end{aligned}$$

for any $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$. On the other hand $q_3\big(f_1(v_1)f_2(v_2), f_3(v_3)\big) = q_3\big(f_3(v_1v_2), f_3(v_3)\big)$ for all v_3 in V_3 if and only if $f_1(v_1)f_2(v_2) = f_3(v_1v_2)$, and this proves the statement. ♠

We leave it to the reader as an exercise to identify the restricted automorphism groups for each of the elementary examples of 10.6.

Proposition 10.12. *The image of the natural embedding*

$$\mathrm{Spin}_8 \subseteq \mathrm{SO}_8 \times \mathrm{SO}(S^+, \beta) \times \mathrm{SO}(S^-, \beta)$$

given by $(\rho, \sigma^+, \sigma^-)$ is $\mathrm{Aut}^0(\mathbb{C}^8, S^+, S^-, \Phi)$.

Proof. For each $a \in \mathrm{Spin}_8$, the map $\rho(a) \in \mathrm{SO}_8$ is defined as $\rho(a)v = ava^{-1}$, with the product being the Clifford product; while $\sigma^+(a)x$ and $\sigma^-(a)y$ are respectively ax and ay, the product being given by the structure of $\mathrm{C}^+(\mathbb{C}^8, q)$-module on S^+ and S^-. Hence

$$\begin{aligned}\big(\rho(a)v\big)\big(\sigma^+(a)x\big) &= ava^{-1}ax\\ &= a(vx)\\ &= \sigma^-(a)(vx);\end{aligned}$$

this shows that $\big(\rho(a), \sigma^+(a), \sigma^-(a)\big)$ is a restricted automorphism of $(\mathbb{C}^8, S^+, S^-, \Phi)$. This gives an embedding of Spin_8 into $\mathrm{Aut}^0(\mathbb{C}^8, S^+, S^-, \Phi)$.

Let us prove that this inclusion is an equality. First of all, let us check that $\mathrm{Aut}^0(\mathbb{C}^8, S^+, S^-, \Phi)$, which is a priori a subgroup of $\mathrm{O}_8 \times \mathrm{O}(S^+, \beta) \times \mathrm{O}(S^-, \beta)$, is in fact contained inside $\mathrm{SO}_8 \times \mathrm{O}(S^+, \beta) \times \mathrm{O}(S^-, \beta)$.

The point is the following. The element $f \in \mathrm{O}_8$ induces an automorphism of $\mathrm{C}(\mathbb{C}^8, q)$, as in the discussion preceding the statement of Proposition 8.9. The direct sum $S^+ \oplus S_- = \bigwedge^\bullet W$ is a module over $\mathrm{C}(\mathbb{C}^8, q)$-module; the equalities $f_1(v_1)f_2(v_2) = f_3(v_3)$ and $f_1(v_1)f_3(v_3) = f_2(v_2)$ tell us that $f_2 \oplus f_3\colon V_2 \oplus V_3 \to V_2 \oplus V_3$ induce an isomorphism $\bigwedge^\bullet W \simeq \big(\bigwedge^\bullet W\big)^{f_1}$ of $\mathrm{C}(\mathbb{C}^8, q)$-modules. Hence f_2 and f_3 give isomorphisms of $\mathrm{C}^+(\mathbb{C}^8, q)$-modules $S^+ \simeq (S^+)^{f_1}$ and $S^- \simeq (S^-)^{f_1}$; and Proposition 8.9 implies that $f_1 \in \mathrm{SO}_8$, as claimed.

The kernel of $\rho\colon \mathrm{Spin}_8 \to \mathrm{SO}_8$ is $\{\pm 1\}$; hence to show that the embedding is an equality it suffices to prove that the kernel of the homomorphism

$$\mathrm{Aut}^0(\mathbb{C}^8, S^+, S^-, \Phi) \longrightarrow \mathrm{SO}_8$$

that sends (f_1, f_2, f_3) into f_1 has order 2. Let $(\mathrm{id}_{\mathbb{C}^8}, f, g)$ be an element of this kernel. Then we have $g(vx) = vf(x)$ for any $v \in \mathbb{C}^8$ and $x \in S^+$; and this implies that $f \oplus g\colon S^+ \oplus S^- \to S^+ \oplus S^-$ is a homomorphism of $\mathrm{C}(\mathbb{C}^8, q)$-modules. Since $S^+ \oplus S^- = \bigwedge^\bullet W$ is a simple $\mathrm{C}(\mathbb{C}^8, q)$-module, (f, g) must be a scalar; that is, both f and g are scalars, and they are equal. Since they are orthogonal they must be ± 1; this concludes the proof. ♠

There is a more general notion of automorphism of a triality, in which the factors are allowed to be permuted.

Definition 10.13. Let $\tau = (V_1, V_2, V_3, \Phi)$ be a triality. An *automorphism* of τ is a quadruple (f_1, f_2, f_3, s), where $s \in S_3$, and $f_i\colon V_i \to V_{s(i)}$ is an isometry, such that

$$\Phi\big(f_{s^{-1}(1)}(v_{s^{-1}(1)}), f_{s^{-1}(2)}(v_{s^{-1}(2)}), f_{s^{-1}(3)}(v_{s^{-1}(3)})\big) = \Phi(v_1, v_2, v_3)$$

for any $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$.

The composition of automorphisms is defined by the formula

$$(f_1, f_2, f_3, s) \circ (g_1, g_2, g_3, t) = (f_{t(1)} \circ g_1, f_{t(2)} \circ g_2, f_{t(3)} \circ g_3, st).$$

If $f \overset{\mathrm{def}}{=} (f_1, f_2, f_3, s)$ is an automorphism, this extends to a linear automorphisms of $V_1 \oplus V_2 \oplus V_3$ by the formula

$$f(v_1, v_2, v_3) = \big(f_{s^{-1}(1)}(v_{s^{-1}(1)}), f_{s^{-1}(2)}(v_{s^{-1}(2)}), f_{s^{-1}(3)}(v_{s^{-1}(3)})\big).$$

Conversely, this automorphism f determines (f_1, f_2, f_3, s); we will denote an automorphism of τ by specifying f.

We leave it to the reader to check that with the product thus defined, the automorphisms of τ form a group, called the *automorphism group of* τ, denoted by $\mathrm{Aut}(\tau)$. If (f_1, f_2, f_3, s), the f_i define an orthogonal map of $V_1 \oplus V_2 \oplus V_3$ into itself, by the formula above; this gives an injective automorphism of $\mathrm{Aut}(\tau)$ into the orthogonal group $\mathrm{O}(V_1 \oplus V_2 \oplus V_3)$.

There is a natural homomorphism $\mathrm{Aut}(\tau) \to \mathrm{S}_3$, sending (f_1, f_2, f_3, s) into s; its kernel is canonically isomorphic to the restricted automorphism group $\mathrm{Aut}^0(\tau)$.

Here is the main result of our treatment.

Theorem 10.14. *The homomorphism* $\mathrm{Aut}(\tau) \to \mathrm{S}_3$ *is a split surjection.*

That is, there is a group homomorphism $\mathrm{S}_3 \to \mathrm{Aut}(\tau)$ whose composite with $\mathrm{Aut}(\tau) \to \mathrm{S}_3$ is the identity. Once such a splitting is chosen, it induces a right action of S_3 by conjugation on the normal subgroup $\mathrm{Aut}^0(\tau) \subseteq \mathrm{Aut}(\tau)$. The group $\mathrm{Aut}^0(\tau)$ is a subgroup of $\mathrm{O}(V_1, q_1) \times \mathrm{O}(V_2, q_2) \times \mathrm{O}(V_3, q_3)$; hence V_1, V_2 and V_3 are representations of $\mathrm{Aut}^0(\tau)$.

If $f \in \mathrm{Aut}(\tau)$ and W is a representation of $\mathrm{Aut}^0(\tau)$, we denote by W^f the representation of $\mathrm{Aut}^0(\tau)$ on the same space W given by the formula $(h, w) \mapsto (f^{-1}hf)w$. I claim that if $f \in \mathrm{Aut}(\tau)$ maps to $s \in \mathrm{S}_3$, then for each $i = 1, 2$ or 3, the map $f_i\colon V_i \to V_{s(i)}$ gives an isomorphism of representations of V_i^f with $V_{s(i)}$. Let $h = (h_1, h_2, h_3) \in \mathrm{Aut}^0(\tau)$; then we have

$$\begin{aligned} h\big(f_i v\big) &= h_{s(i)} f_i(v) \\ &= f_i\big((f_i^{-1} h_{s(i)} f_i) v\big) \\ &= f_i\big((f^{-1} h f) v\big) \end{aligned}$$

for any $v \in V_i$. Applying this to $\mathrm{Spin}_8 = \mathrm{Aut}^0(\mathbb{C}^8, S^+, S^-)$ we obtain that the right action of R_3 acts as the symmetric group on the isomorphism classes of the representations $\mathbb{C}^8$, S^+ and S^-, as claimed.

Proof. To obtain the isomorphism, choose two vectors $u_1 \in V_1$ and $u_2 \in V_2$ with $q_1(v_1) = q_2(v_2) = 1$. Set $u_3 \overset{\text{def}}{=} u_1u_2 \in V_3$; we have $q_3(v_3) = q_1(v_1)q_2(v_2) = 1$.

Notice that we have

$$\begin{aligned} u_1u_3 &= u_1(u_1u_2) \\ &= q_1(u_1)u_2 \\ &= u_2 \end{aligned}$$

by Lemma 10.9. Analogously, if $\{i, j, k\} = \{1, 2, 3\}$ we have $u_iu_j = u_k$. Thus we can start from any two u_i and u_j with $q_i(v_i) = q_j(v_j) = 1$, and obtain the third as $u_k = u_iu_j$.

For each i, we obtain isometries $u_i- \colon V_j \to V_k$ and $u_i- \colon V_k \to V_j$, defined by $v \mapsto u_iv$; again because of Lemma 10.9, these are the inverse of each other. We start with three lemmas.

We need to compare the composition

$$V_i \xrightarrow{u_k-} V_j \xrightarrow{u_i-} V_k$$

with

$$u_j- \colon V_i \longrightarrow V_k.$$

Denote by $\mathrm{R}_{u_i} \colon V_i \to V_i$ the reflexion along the hyperplane orthogonal to u_i, defined by the equality

$$\mathrm{R}_{u_i}v_i = v_i - 2q_i(u_i, v_i)u_i.$$

Lemma 10.15. *For each $v_i \in V_i$ we have*

$$u_i(u_kv_i) = u_j\bigl(-\mathrm{R}_{u_i}v_i\bigr).$$

Proof. By Lemma 10.9 we have

$$\begin{aligned} u_i(u_kv_i) &= -v_i(u_iu_k) + 2q_i(u_i, v_i)u_k \\ &= -u_jv_i + 2q_i(u_i, v_i)u_k \\ &= u_j\bigl(-v_i + 2q_i(u_i, v_i)u_i\bigr) \\ &= u_j\bigl(-\mathrm{R}_{u_i}v_i\bigr). \end{aligned}$$

♠

Lemma 10.16. *If $\{i, j, k\} = \{1, 2, 3\}$, $v_j \in V_j$ and $v_k \in V_k$, then*

$$(u_iv_k)(u_iv_j) = -\mathrm{R}_{u_i}(v_jv_k) \in V_i.$$

Proof. Assume, for example, that $(i, j, k) = (1, 2, 3)$. By Lemma 10.9, we have

$$\begin{aligned} (u_1v_3)(u_1v_2) &= -v_2\bigl((u_1v_3)u_1\bigr) + 2q_2(u_1v_3, v_2)u_1 \\ &= -v_2v_3 + 2\Phi(u_1, v_2, v_3)u_1 \\ &= -v_2v_3 + 2q_1(u_1, v_2v_3)u_1 \\ &= -\mathrm{R}_{u_1}(v_2v_3). \end{aligned}$$

♠

Lemma 10.17. *If* $\{i, j, k\} = \{1, 2, 3\}$ *and* $v_j \in V_j$*, we have*

$$u_i(\mathrm{R}_{u_j} v_j) = \mathrm{R}_{u_k}(u_i v_j) \in V_k.$$

Proof. This is true because $u_i - \colon V_j \to V_k$ is an orthogonal map and carries u_j into u_k. ♠

We will produce a copy of S_3 inside $\mathrm{Aut}(\tau)$ by lifting the three transpositions $(2\,3)$, $(3\,1)$ and $(1\,2)$. For each $i = 1$, 2 or 3, consider the automorphism σ_i of $V_1 \oplus V_2 \oplus V_3$ that exchanges V_j and V_k by mutiplying by u_i, while it acts on V_i as $-\mathrm{R}_{u_i}$. Thus, for example, we have

$$\sigma_1(v_1, v_2, v_3) = \bigl(-\mathrm{R}_{u_1} v_1, u_1 v_3, u_1 v_2\bigr).$$

I claim that σ_i is in $\mathrm{Aut}(\tau)$. Consider for example the case of σ_1: we need to prove the identity

$$\Phi\bigl(-\mathrm{R}_{u_1} v_1, u_1 v_3, u_1 v_2\bigr) = \Phi(v_1, v_2, v_3)$$

for any $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$. By Lemma 10.16, we have

$$\begin{aligned}
\Phi\bigl(-\mathrm{R}_{u_1} v_1, u_1 v_3, u_1 v_2\bigr) &= q_1\bigl(-\mathrm{R}_{u_1} v_1, (u_1 v_3)(u_1 v_2)\bigr)\\
&= q_1\bigl(-\mathrm{R}_{u_1} v_1, -\mathrm{R}_{u_1}(v_2 v_3)\bigr)\\
&= q_1(v_1, v_2 v_3)\\
&= \Phi(v_1, v_2, v_3)
\end{aligned}$$

as claimed.

Each σ_i maps to the transposition $(j\,k)$ in S_3; therefore the subgroup generated by the σ_i surjects onto S_3. We need to show that it is in fact isomorphic to S_3. For this, it is enough to show that the σ_i satisfy the following relations:

(a) $\sigma_1^2 = \sigma_2^2 = 1$ and
(b) $\sigma_1 \sigma_2 \sigma_1 = \sigma_3 = \sigma_2 \sigma_1 \sigma_2$;

for it is well known, and easy to show, that the group generates by three generators σ_1, σ_2 and σ_3 satisfying the relations above is in fact S_3.

It is immediate to prove that $\sigma_i^2 = 1$. Let us compute $\sigma_2 \sigma_1$. We have

$$\begin{aligned}
\sigma_2 \sigma_1(v_1, v_2, v_3) &= \sigma_2(-\mathrm{R}_{u_1} v_1, u_1 v_3, u_1 v_2)\\
&= \bigl(u_2(u_1 v_2), -\mathrm{R}_{u_2}(u_1 v_3), -u_2(\mathrm{R}_{u_1} v_1)\bigr)\\
&= \bigl(-u_3(\mathrm{R}_{u_2} v_2), -\mathrm{R}_{u_2}(u_1 v_3), -u_2(\mathrm{R}_{u_1} v_1)\bigr)\\
&= \bigl(-u_3(\mathrm{R}_{u_2} v_2), -u_1(\mathrm{R}_{u_3} v_3), -u_2(\mathrm{R}_{u_1} v_1)\bigr)
\end{aligned}$$

because of Lemmas 10.15 and 10.17. Hence

$$\begin{aligned}
\sigma_1 \sigma_2 \sigma_1(v_1, v_2, v_3) &= \sigma_1\bigl(-u_3(\mathrm{R}_{u_2} v_2), -u_1(\mathrm{R}_{u_3} v_3), -u_2(\mathrm{R}_{u_1} v_1)\bigr)\\
&= \bigl(\mathrm{R}_{u_1}(u_3(\mathrm{R}_{u_2} v_2)), -u_1(u_2(\mathrm{R}_{u_1} v_1)), -u_1(u_1(\mathrm{R}_{u_3} v_3))\bigr)\\
&= \bigl(u_3(\mathrm{R}_{u_2}^2 v_2), u_3(\mathrm{R}_{u_1}^2 v_1), -\mathrm{R}_{u_3} v_3)\bigr)\\
&= \bigl(u_3 v_2, u_3 v_1, -\mathrm{R}_{u_3} v_3)\bigr)\\
&= \sigma_3(v_1, v_2, v_3).
\end{aligned}$$

The proof that $\sigma_2 \sigma_1 \sigma_2 = \sigma_3$ is obtained by exchanging the indices 1 and 2 in the formulas above.

♠

APPENDIX A. WEDDERBURN THEORY

We fix a field K; all algebras will be finite algebras over K, and all modules will be finitely generated (or, equivalently, will be finite-dimensional vector spaces over K).

Let A be a K-algebra, M an A-module. We will write ${}_AM$ or M_A to indicated whether M is a left of right A-module. If we write simply "a module", this module will be indifferently left or right.

We will also follow the following convention, which, for very good reasons, is standard in non-commutative algebra: if M is a right A-module, $\mathrm{End}_A(M_A)$ will be the ring of endomorphism of M, written as usual on the left, and composed according to the usual rule. Then M will be a both a right A-module and a left $\mathrm{End}_A(M_A)$-module, and the two structures as linked as follows: if $f \in \mathrm{End}_A(M_A)$, $x \in M$ and $a \in A$, then $f(xa) = (fx)a$ (one says that M is an $\big(\mathrm{End}_A(M_A) - A\big)$-*bimodule)*. But if M is a left A-module, then we write the endomorphisms on the *right*, and the composition is defined by the opposite rule: fg is composed by first applying f then g, so that $x(fg) = (xf)g$ for any $x \in M$. Then M is a left A-module and a right $\mathrm{End}_A({}_AM)$-module, and we have $(ax)f = a(xf)$ for any $a \in A$, $x \in M$ and $f \in \mathrm{End}_A({}_AM)$ (M is an $(A - \big(\mathrm{End}_A(M_A)\big)$-bimodule)).

A *division algebra* is non-zero K-algebra in which every non-zero element is invertible. If D is a division algebra, then D contains K in its center.

Proposition A.1. *Suppose that K is algebraically closed. Then K is the only finite division algebra over K, up so isomorphism.*

Proof. If $x \in D$, then the subring $K[x] \subseteq D$ is a finite field extensions of K. It follows that $D = K$.

An alternate argument is as follows: for each $u \in D$, consider the K-linear operator $D \to D$ defined by $x \mapsto ux$. Since K is algebraically closed and D is a vector space of finite positive dimension, this operator has an eigenvalue $a \in K$. Hence multiplication by $u - a$ on D is not injective; but then $u - a = 0$, and $u \in K$. ♠

A right A-module M_A is *free* if it has a *basis*, a sequence of elements $e_1, \dots, e_n$ such that every element $x \in M$ can be written uniquely in the form

$$x = e_1a_1 + \cdots + e_na_n$$

with $a_1, \dots, a_n$ in A. Equivalently, M_A is free if it isomorphic to the free right A-module A^n_A. We will write vectors in A^n_A as column vectors: then then endomorphism ring $\mathrm{End}_A(A^n_A)$ is the matrix algebra $\mathrm{M}_n(A)$, with the action given matrix multiplication of a square matrix by a column vector, in the usual fashion (the usual argument for fields will work; or see Lemma A.13).

Notice that the dimension of A^n as a vector space over K is $n \dim_K A$: hence if $A \neq 0$ the cardinality of a basis of a free module is uniquely determined.

Proposition A.2. *A module over a division algebra is free.*

One takes one of the standard proofs of this fact for fields and check that it works for division rings.

Definition A.3. Let A be a K-algebra. A module M over A is *simple* (or *irreducible*) if it not 0, and has no non-zero proper submodule.

Clearly, M is simple if and only if it it not zero, and is generated by any of its non-zero elements.

Proposition A.4 (Schur's lemma)**.** *Let V be a simple A-module.*

(a) *A morphism of left A-modules $f\colon V \to M$ is either 0 or injective. If M is also simple, then f is either 0 or an isomorphism.*
(b) *The endomorphism algebra $\mathrm{End}_A(V)$ is a division algebra.*
(c) *If K is algebraically closed, then $\mathrm{End}_A(V) = K$.*

Proof. The kernel of f is a submodule of M and the image of f is a submodule of N: this proves part (a). Part (b) follows from part (a), and part (c) follows from part (b) and Proposition A.1. ♠

Corollary A.5. *The endomorphims algebra $\mathrm{End}_A({}_AM)$ of a simple A-module is a division algebra.*

Proposition A.6. *Let M be a module over A. The following conditions are equivalent.*

(a) *M is a sum of simple submodules.*
(b) *M is a direct sum of simple submodules.*
(c) *Every submodule of M has a* complement, *that is, if N is a submodule of M, there exists another submodule $N' \subseteq N$ such that $M = N \oplus N'$.*

Definition A.7. If the equivalent conditions of Proposition A.6 are satisfied, we say that M is *semisimple*.

Proof. Clearly (b) implies (a).

Let us prove that (a) implies (c). Let M be a sum of simple modules, and let N be a submodule of M. Choose a submodule N' which has maximal dimension among those with $N \cap N' = 0$; then the sum $N + N'$ is direct. I claim that $N + N' = M$. If not, there would exist a simple module V of M that is not contained in $N + N'$. The intersection $V \cap (N + N')$ is a proper submodule of V, so it is 0. This implies that $N \cap (N' + V) = 0$, which is absurd, because $N' + V$ contains V properly.

We conclude by showing that (c) implies (b). Let N be a submodule of M that has maximal dimension among all those that direct sums of simple submodules, say $N = V_1 \oplus \cdots \oplus V_k$: I claim that $N = M$. If not, let V_{k+1} be a submodule of N' that has minimal dimension among all the non-zero submodules of N'; clearly V_{k+1} is simple. We have $N \cap V_{k+1} = 0$, which implies that the sum $N + V_{k+1}$, which contains N properly, is a direct sum $V_1 \oplus \cdots \oplus V_k \oplus V_{k+1}$. This is absurd, because N was supposed to be maximal. ♠

Corollary A.8. *If M is a semisimple A-module, every quotient and every submodule of M is semisimple. Furthermore, every simple module that is contained in a quotient or a submodule of M is also contained in M.*

Proof. Let $\rho\colon M \to N$ be a quotient of M, and write M as a sum of simple submodules V_i. The image $\rho(V_i)$ is either 0 or isomorphic to V_i, by Proposition A.4 (a), and N the sum of the $\rho(V_i)$. This implies that it semisimple.

On the other hand Proposition A.6 (c) implies that every submodule of M is isomorphic to a quotient of M. ♠

Let $D_1, \ldots, D_r$ be division algebras over K, $n_1, \ldots, n_r$ be positive integers; we will characterize left A-modules over the algebra

$$A = \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_r}(D_r)$$

(similar arguments would work for right modules).

For each i, set $V_i = D_i^{n_i}$; let A act on V_i on the left through the i^{th} projection. It is easy to see that any non-zero element of V_i can be completed to a basis of V_i, and that, as a consequence, given any two elements v and w of V_i with $v \neq 0$ there exists a matrix $\alpha \in \mathrm{M}_{n_i}(D_i)$ such that $\alpha v = w$ (this can be proved as for fields). This implies that V_i is generated by all non-zero elements, hence it is simple. It is easy to verify that the ring $\mathrm{End}_A(V_i)$ coincides with D_i.

Clearly V_i is not isomorphic to V_j for $i \neq j$, for example because the annihilators are different.

Proposition A.9. *As a left A-module, A is isomorphic to $V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$.*

Proof. Each ring $\mathrm{M}_{n_i}(D_i)$ is isomorphic to $V_i^{n_i}$, the isomorphism being given by writing a matrix as the sequence of its n_i column vectors. If $e_1, \ldots, e_{n_i}$ is the canonical basis of D_i as a right D_i-module, this isomorphism is obtained by sending α into $(\alpha e_1, \ldots, \alpha e_{n_i})$. From these we obtain the desidered isomorphism of left A-modules

$$A \simeq V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}.$$

♠

Theorem A.10. *Every module over A is isomorphic to a unique module of the form $V_1^{d_1} \oplus \cdots \oplus V_r^{d_r}$, for a uniquely determined sequence of non-negative integers $d_1, \ldots, d_r$.*

In particular, the V_i are the only simple modules over A.

Proof. Every left module over A, which is finitely generated by hypothesis, is a quotient of A^m for some non-negative integer m: hence by Corollary A.8 every module is a sum of copies of the V_i.

Uniqueness is easily seen from the fact that $\mathrm{Hom}_A(V_i, V_j) = 0$ for $i \neq j$, because of Proposition A.4 (a), so

$$\mathrm{Hom}_A(V_i, V_1^{d_1} \oplus \cdots \oplus V_r^{d_r}) = \mathrm{Hom}_A(V_i, V_i)^{d_i}.$$ ♠

Definition A.11. A K-algebra is *semisimple* if it it semisimple as a left A-module.

So finite products of matrix algebras over division algebras are semisimple. These are in fact the only examples.

Let A be a semisimple algebra: let $V_1, \ldots, V_r$ be pairwise non-isomorphic simple left modules over A, such that, as a left A-module, ${}_AA$ is isomorphic to $V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$. For each i set $D_i = \mathrm{End}_A(V_i)$; by Proposition A.4 (b), D_i is a division algebra.

Theorem A.12 (Wedderburn)**.** *As an algebra, A is isomorphic to the product $\mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_r}(D_r)$. Furthermore each n_i coincides with the dimension of V_i as a right vector space over D_i.*

Proof. Right multiplication defines an action of A on itself, that commutes with left multiplication; thus we get a homomorphism of algebras $A \to \mathrm{End}_A({}_AA)$, that is an isomorphism. On the other hand an endomorphism of $A = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$ as a left module will carry each $V_i^{n_i}$ to itself, because $\mathrm{Hom}_A(V_i, V_j) = 0$ for $i \neq j$; hence as an algebra A is isomorphic to $\mathrm{End}_A(V_1^{n_1}) \times \cdots \times \mathrm{End}_A(V_r^{n_r})$. We get the isomorphism from the following Lemma.

Lemma A.13. *Let M be a module over an algebra A, n a positive integer. Then* $\operatorname{End}_A(M^n)$ *is isomorphic as a K-algebra to* $\mathrm{M}_n\bigl(\operatorname{End}_A(M)\bigr)$.

Proof. Given an endomorphism $f\colon M^n \to M^n$ and two indiced i and j between 1 and n, let $f_{ij}\colon M \to M$ be the endomorphism that sends $x \in M$ into the j^{th} component of $f(\xi)$, where $\xi \in M^n$ is the vector that has x at the i^{th} place and 0 everywhere else. Then (f_{ij}) is a matrix in $\mathrm{M}_n\bigl(\operatorname{End}_A(M)\bigr)$; we leave it to the reader to check that by sending f into (f_{ij}) one obtains an isomorphism of rings. ♠

For the last statement, we have seen that the simple modules over A are $D_1^{n_1}$, …, $D_r^{n_r}$; by looking at annihilators we see that each V_i must be isomorphic to $D_i^{n_i}$. ♠

Corollary A.14. *Suppose that K is algebraically closed, and let $V_1, \dots, V_r$ be pairwise non-isomorphic simple left A-modules. Then the induced homomorphism*

$$A \longrightarrow \operatorname{End}_K(V_1) \times \cdots \times \operatorname{End}_K(V_r)$$

describing the action of A on the V_i is surjective.

Proof. Set $R \overset{\text{def}}{=} \operatorname{End}_K(V_1) \times \cdots \times \operatorname{End}_K(V_r)$. We may substitute A with its image in R. Call n_i the dimension of V_i over K; then we know that we have an isomorphism of ${}_RR$ with $V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$; hence ${}_AA$ is a submodule of $V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$. By Schur's Lemma (Proposition A.4 (c)) we have that $K = \operatorname{End}_A(V_i)$ for each i; by Wedderburn's theorem A.12 we get that A is isomorphic to $\mathrm{M}_{n_1}(K) \times \cdots \times \mathrm{M}_{n_r}(K)$. By comparing dimensions we see that $A = R$, as required. ♠

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